

WAVE-HEIGHT DISTRIBUTIONS AND NONLINEAR EFFECTS

M. Aziz Tayfun

Civil Engineering Department
Kuwait University, Kuwait
aziztayfun@usa.net

Francesco Fedele

GEST & GMAO, NASA Goddard Space Flight Center
Greenbelt, MD, USA.
fedele@gmao.gsfc.nasa.gov

ABSTRACT

Theoretical distributions for describing the crest-to-trough heights of linear waves are reviewed briefly. To explore the effects of nonlinearities, these and two approximations that follow from the Tayfun [28] model are generalized to nonlinear waves via the second-order quasi-deterministic model of Fedele and Arena [7]. The potential utility of Gram-Charlier type approximations [17, 18, 29, 31] in representing the statistics of nonlinear wave heights is also explored. All models and a fifth-order Stokes-Rayleigh type model recently proposed by Dawson [5] are compared with linear and nonlinear waves simulated from the JONSWAP spectrum representative of long-crested extreme seas, and also with observational data gathered during two severe storms in the North Sea. The results indicate first that nonlinearities do not appear to affect the crest-to-trough wave heights significantly. Most models and their nonlinear extensions yield similar and reasonable predictions of the data trends observed. The present comparisons do not confirm the efficacy of Gram-Charlier type approximations in modeling the statistics of unusually large wave heights.

INTRODUCTION

Current interest in the mechanics and statistics of large waves necessitates a re-examination of various theoretical forms for describing the distribution of wave heights [6-8, 11-14, 17, 18, 20-24, 32, 33]. Over the years, a variety of numerical and analytic wave-height distributions have been proposed. This study will first consider an initial short list of just three analytical models due to Naess [19], Boccotti [1] and Tayfun [25, 28]. Of these, Naess's model (N) has received wide popularity. This is justifiably so due to its simple normalized form despite the fact that previous comparisons [28] and those to be presented here indicate that N consistently underestimates the observed wave heights slightly. In contrast, Tayfun's model

(T) for large wave heights appears more complicated. However, Boccotti's model (B) is just about as simple as N, but has not received the attention that it probably deserves. Thus, one of the present objectives is to review these models briefly, including two obvious and simple approximations (T1 & T2) that follow from T. All are then compared with simulated waves and 9-h measurements gathered from the TERN platform in 167 m water depth in the northern North Sea during a severe storm in January, 1993 [10].

Since large waves are nonlinear, the linear models considered are subsequently modified to include the effects of second-order nonlinearities, using the extension of Boccotti's [1] linear quasi-deterministic theory to second-order waves by Fedele and Arena [7]. These and the fifth-order Stokes-Rayleigh model of Dawson [5] are then compared with simulated nonlinear waves and the TERN data.

Although most oceanic data analyzed in the past as well as the present TERN measurements clearly indicate that wave heights are not noticeably affected by nonlinearities, the analyses and comparisons in [12, 14, 17, 18, 20, 23] and several other recent studies suggest that the occurrence of extreme wave heights systematically larger than those typically predicted with the conventional probability laws can be explained in terms of third-order free-wave interactions and the attendant fourth-order normalized cumulants of the sea surface. Thus, to include the potential effects of such nonlinearities, two approximate Gram-Charlier (G-C) distributions are also considered, guided by the results in Tayfun and Lo [29] and Mori and Janssen [18]. The potential effects of high-order nonlinearities and issues related to the sampling properties of probability estimates are then examined to a limited extent by way of comparisons of the G-C type approximations to a second data set comprising five 20-min measurements gathered during severe weather conditions from

the Draupner platform in the central North Sea in 70 m water depth in January, 1995 [11].

LINEAR MODELS

The exceedance distribution functions (EDF) of present interest and applicable to the crest-to-trough wave height, say H , scaled with the sea-surface rms σ have the general form:

$$E = \Pr\{H/\sigma > h\} = c_0 f(h) \exp(-c_1 h^2) \quad (1)$$

The corresponding probability density function (PDF) easily follows from $-dE/dh$. The parameters c_0 , c_1 and the function f are summarized all in Table 1 for N, B, T, T1, T2 and the conventional Rayleigh (R) EDF. To elaborate these further, let m_j , $\rho(\tau)$ and $\rho''(\tau)$ represent, respectively, the ordinary spectral moments and the auto-correlation coefficient functions of the surface elevation and its temporal derivative. The upper (+) and lower (-) envelopes of $\rho(\tau)$ are $\pm r(\tau) = \pm \sqrt{\rho^2 + \hat{\rho}^2}$, where $\hat{\rho}$ is the Hilbert transform of ρ [28]. On this basis, $r = r(0.5T_m)$, $a = \rho(\tau^*)$ and $b = \rho''(\tau^*)$, where $T_m = 2\pi(m_0/m_1)$, and $\tau^* \equiv$ time lag at which the first minimum of ρ occurs [2, 19]. Obviously, all the preceding parameters can be estimated either from a time series of the surface or, somewhat more accurately and just as simply, from the associated frequency spectrum [2, 9, 28].

It is noticed that only N, T2 and R are properly normalized to unity at $h = 0$. Others are not since they are derived using the asymptotic behavior of either the surface itself, as in B, or its envelope, as in T and T1 both. So, the non-normalized EDFs in the table are valid over the range of relatively large waves whereas the normalized ones are valid for $h \geq 0$. Note further that T1 corresponds to the lower bound EDF in [28]. Thus, T2 represents a normalized approximation to that, as N to B. As the spectrum bandwidth approaches zero, then $\rho \rightarrow -a \rightarrow b \rightarrow 1$, and all EDFs assume the same limit form of R. In general, r is similar to, but always slightly larger than $-a$, leading to the result that $T2 \text{ EDF} \geq N \text{ EDF}$ invariably.

SECOND-ORDER WAVES

Suppose that the surface elevation measured from the mean sea level at a fixed point and scaled with σ is of the usual second-order form $\eta = \eta_1 + \eta_2$, and that a wave crest occurs at time $t = t_0$. Defining $\xi_1 \equiv \eta_1(t_0)$ as the linear component of that crest, the conditional mean of $\eta_1(t_0 + \tau)$, given $\eta_1(t_0) = \xi_1 \gg 1$, is described by [2, 15, 16, 21]

Model	c_0	c_1	$f(h)$
N	1	$\frac{1}{4(1-a)}$	1
B	$\frac{1+b}{\sqrt{2b(1-a)}}$	$\frac{1}{4(1-a)}$	1
T	$\sqrt{\frac{1+r}{2r}}$	$\frac{1}{4(1+r)}$	$1 + \frac{1-r^2}{4r h^2}$
T1	$\sqrt{\frac{1+r}{2r}}$	$\frac{1}{4(1+r)}$	1
T2	1	$\frac{1}{4(1+r)}$	1
R	1	$\frac{1}{8}$	1

Table 1. Parameter definitions in different EDF models.

$$\langle \eta_1(t_0 + \tau) | \eta_1(t_0) = \xi_1 \rangle = \xi_1 \rho_1(\tau) \quad (2)$$

where $\rho_1 = \langle \eta_1(t_0 + \tau) \eta_1(t_0) \rangle = \rho = \langle \eta(t_0 + \tau) \eta(t_0) \rangle$, correct to first-order in wave steepness and consistent the second-order theory. The extension of this result to second-order waves by Fedele and Arena [7] gives the conditional mean of η as

$$\langle \eta(t_0 + \tau) | \eta(t_0) = \xi \rangle = \xi \rho(\tau) + \frac{1}{6} \xi_1^2 \lambda(\tau) \quad (3)$$

where $\xi \equiv$ second-order nonlinear crest height at t_0 ; and,

$$\begin{aligned} \lambda(\tau) &= 3 \langle \eta_1^2(t_0) \eta(t_0 + \tau) \rangle \\ &= \frac{3}{2\sigma^3} \iint_{\vec{k} \vec{k}'} \psi \psi' [K^+ \cos \phi^+ + K^- \cos \phi^-] dk dk' \end{aligned} \quad (4)$$

with $\vec{k} \equiv$ wave-number vector, $\psi \equiv$ wave-number spectrum, $\phi^\pm = (\omega \pm \omega')\tau$, $\omega \equiv$ angular frequency related to $|\vec{k}|$ via the usual dispersion relationship, and $K^\pm \equiv$ second-order interaction kernels (see e.g. [10]). It turns out that $\lambda(\tau) \leq \lambda(0) = \lambda_3 = \langle \eta^3 \rangle \equiv$ skewness coefficient of η [31, 32]. So, the nonlinear crest height at t_0 follows from Eq. (3) with $\tau = 0$ as

$$\xi = \xi_1 + \frac{1}{6} \lambda_3 \xi_1^2 \quad (5)$$

The wave trough, say, ζ immediately following the crest ξ at t_0 occurs at time $t = t_0 + \tau^*$. From Eq. (3)

$$\zeta = \xi_1 \rho(\tau^*) + \frac{1}{6} \lambda(\tau^*) \xi_1^2 \quad (6)$$

Thus, the crest-to-trough wave height $h \equiv \xi - \zeta$ can be written in the simple quadratic form

$$h = h_1 + \frac{1}{2} \beta h_1^2 \quad (7)$$

where $h_1 = \xi_1(1-a) \equiv$ first-order linear wave height, $a = \rho(\tau^*)$ as before, and

$$\beta = \frac{1}{3} \frac{\lambda_3 - \lambda^*}{(1-a)^2} \quad (8)$$

with $\lambda^* = \lambda(\tau^*)$ for simplicity. For 2D deep-water waves, $K^\pm = \pm |\omega^2 \pm \omega'^2| / g$ so that λ^* as given by Eq. (4) reduces to a double integral of the frequency spectrum, thus simplifying β somewhat [7]. It can also be estimated from a time series of η in a more practical but somewhat approximate fashion.

Strictly, h_1 has the form implied by N only. However, all the models in Table 1 can be used for the linear wave heights just as well, including all Ts if the parameters λ^* and a are replaced by $\lambda(0.5T_m)$ and r , respectively. Numerical computations with various JONSWAP-type spectra show that these modifications introduce negligible errors in the resulting β , and that $\beta \approx 0.003 \sim 0.022$ typically. The larger values represent relatively wide-band spectra that decay $\propto \omega^{-4}$ over high frequencies, and the smaller values imply relatively narrow-band spectra.

It is noticed from Eq. (7) that

$$\frac{h}{h_1} = 1 + \frac{1}{2} \beta h_1 \quad (9)$$

With $\beta \approx 0.003 \sim 0.004$ representing a typical narrow-band case, Eq. (9) gives $1.006 \sim 1.008 \leq h/h_1 \leq 1.036 \sim 1.048$ for $4 \leq h_1 \leq 12$. This result suggests that the effect of second-order nonlinearities is simply rather small for narrow-band waves. The situation is somewhat different for waves with wide-band spectra, as will be shown later. Nonetheless, it is clear at this point that the preceding model coupled with R as a model

for h_1 can not possibly explain unusually large waves attributed to nonlinear interactions of second and higher order.

The EDF of h would follow from Eq. (1) simply as

$$E(h) = E(h_1) = c_0 f(h_1) \exp(-c_1 h_1^2) \quad (10)$$

where $h_1 = (-1 + \sqrt{1 + 2\beta h}) / \beta$. The corresponding PDF is then given by $-(dE/dh_1)(dh_1/dh)$.

DAWSON MODEL

The EDF of scaled nonlinear wave heights in Dawson's fifth-order model (D) [5] is given in the present notation by

$$E(h) = \exp(-2\alpha_D^2 h^2) [1 - \frac{3}{16} \mu_D^2 h^2 - \frac{107}{3072} \mu_D^4 h^4] \quad (11)$$

where $\alpha_D \equiv \sigma / H_s$; $\mu_D = \sigma \bar{\omega}^2 / g \equiv$ steepness parameter; $\bar{\omega} = 2\pi / \langle T \rangle$; and, H_s and $\langle T \rangle$ represent, respectively, the significant wave height and mean zero-up-crossing period derived from actual measurements. These definitions suggest that D can not be used before the latter two statistics are actually estimated from the data itself. So, D is not a predictive model in the same sense as the other models considered here.

COMPARISONS

The simulation of 2D linear deep-water waves uses a modified JONSWAP spectrum of the form

$$S(\omega) = \frac{m_0}{\omega_p} u^{-4} \exp\left(-\frac{5}{4u^4}\right) \gamma^{g(u)} q(u) \quad (12)$$

where $\omega_p \equiv$ spectral-peak frequency, $\gamma \equiv$ peak enhancement factor, $0.2 \leq u = \omega / \omega_p \leq 10$, $g(u) \equiv$ standard exponent function of γ , and $q(u) = 1$ for $u \leq 4$, and $q(u) = (u/4)^{-4}$ for $u > 4$. The 'filter' function $q(u)$ helps simulate second-order waves whose spectrum attenuates as ω^{-4} approximately. This is consistent with the behavior of oceanic spectra and the second-order results in [30].

In the present simulations, $m_0 \cong 10 \text{ m}^2$, $\omega_p = 0.467 \text{ s}^{-1}$ ($T_p \cong 13.44 \text{ s}$) and $\gamma = 3.3$. An ensemble of 40 linear and nonlinear time series sampled at 5 Hz was simulated, following an efficient approach elaborated in [27], but with linear random spectral amplitudes whose rms values mimic the spectral shape in Eq. (12). These yield altogether about 450,000 linear waves first and subsequently 470,000 nonlinear waves. The linear and nonlinear target spectra are compared with the spectra of linear and nonlinear series simulated in Fig. 1.

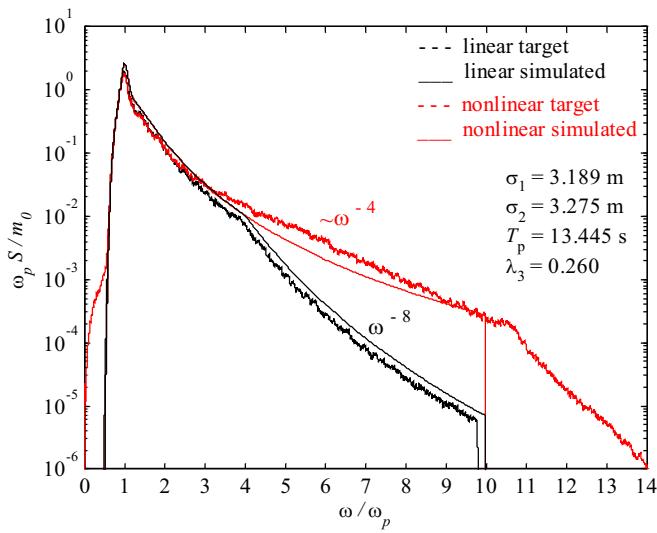


Fig. 1. Targeted and simulated spectra.

Prior to comparisons, consider the order statistics of a set of n independent observations of h with the theoretical PDF and EDF given by p and E , respectively. Rank-order these into the order statistics $h_1 > h_2 > \dots > h_n$. The PDF of h_j is given by (see e.g. [3])

$$p_j(x) = \frac{n!}{(j-1)!(n-j)!} E^{j-1} (1-E)^{n-j} p(x) \quad (13)$$

Regard E now as a random variable, as it would be in estimating it from the order statistics. Since $dE = -p(x)dx$ by definition, the PDF of E follows from the identity $p_E | dE | = p_j | dx |$ as

$$p_E(E) = \frac{n!}{(j-1)!(n-j)!} E^{j-1} (1-E)^{n-j} \quad (14)$$

where $0 \leq E \leq 1$ by definition. The corresponding mean and standard deviation are easily evaluated after some algebra as

$$\langle E \rangle = \frac{j}{n+1} \quad (15)$$

$$\sigma_E = \frac{1}{n+1} \sqrt{\frac{j(n-j+1)}{n+2}} \quad (16)$$

Thus, the coefficient of variation $\delta_E = \sigma_E / \langle E \rangle$ is given by

$$\delta_E = \sqrt{\frac{n-j+1}{j(n+2)}} \quad (17)$$

Notice that $\delta_E \approx \sqrt{1/j}$ for large n , given $j \ll n$. Thus, for the largest few wave heights, the stability of the estimate E is

relatively poor. In particular, δ_E of the largest wave height is 100% since σ_E is as large as E itself. And, it does not get much better for the next few largest waves either. Clearly, the larger the sample population n , the more stable are the estimates of E , but the stability characteristics of the largest group of wave heights, say, the largest five or so remain persistently poor. This result should be of some concern in interpreting the frequency of occurrence associated with just a few exceptionally large waves in oceanic measurements.

A comparison of the theoretical models with the simulated wave heights in the linear case is given in Fig. 2 in a semi-logarithmic form. This figure shows only R, B, T and T1 predictions as it is rather difficult to contrast with clarity all the models in this type of conventional plots. Differences between various models are more easily seen in a form popularized by Forristall [9] and shown in Fig. 3, where the ratio h/r is plotted as a function of E , with $r = \sqrt{-8 \ln(E)}$ representing the wave height that would be predicted by the Rayleigh law at the same E level. Notice that Fig. 2 includes the comparisons for 453,416 wave heights derived from 40 independent simulations pooled together and also 11,385 wave heights from the series containing the largest wave height to demonstrate the effect of the sample size n on both the estimates of E and their stability. The latter are indicated in the figure as bands of $E \pm \sigma_E$ in the same colors as the simulated data. The high variability in the estimates of E for the largest waves is obvious. This aside, the simulated data trends follow the B, T and T1 predictions quite well, as more clearly seen in Fig. 3. Among all models, B appears to describe the relatively large wave heights most accurately, except toward the very extreme tail where T and T1

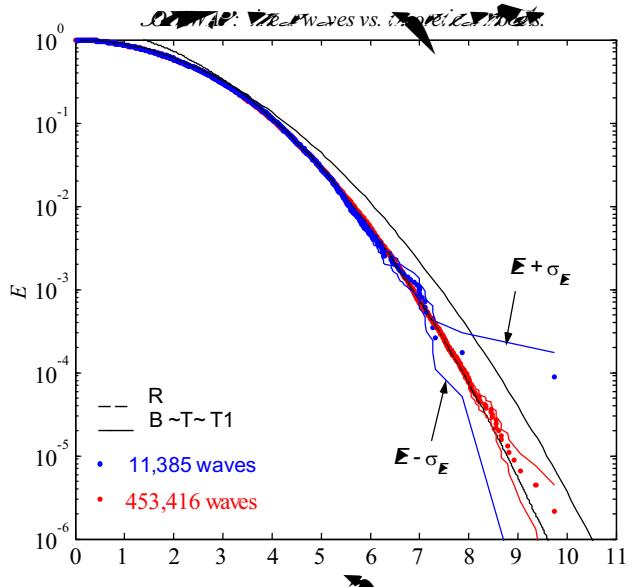


Fig. 2. Linear wave heights, theoretical predictions, and stability of E versus sample size n .

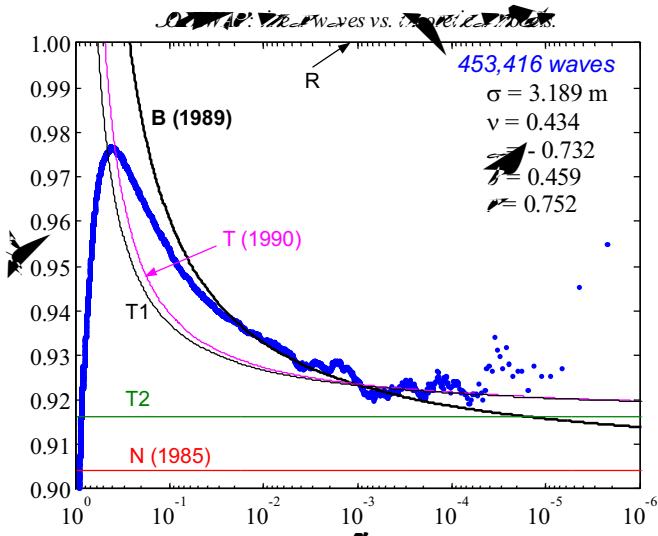


Fig. 3. Linear wave heights vs. theoretical models.

coalesce and represent the data somewhat better. R over predicts the simulated wave heights by 7-8%, as is often noted. In contrast, N under predicts the larger waves by about 2%. So does T2 also but with errors less than 1%. In general, all the models except R appear to describe the simulated data well over the high wave range with maximum errors of about 2% or less.

Linear and nonlinear wave heights are compared in Fig. 4 with the predictions from Eq. (10) appropriate to the second-order quasi-deterministic (Q-D) extensions of B, T, T1, T2 and N. The latter are designated as B Q-D, T Q-D, etc. to differentiate them from their linear limits. The same figure also includes D of Eq. (11) with $\alpha_D = 0.263$ and $\mu_D = 0.167$ estimated from the simulated nonlinear data. Evidently, second-order nonlinearities tend to increase the relative frequency of larger waves, and also amplify their heights somewhat. This

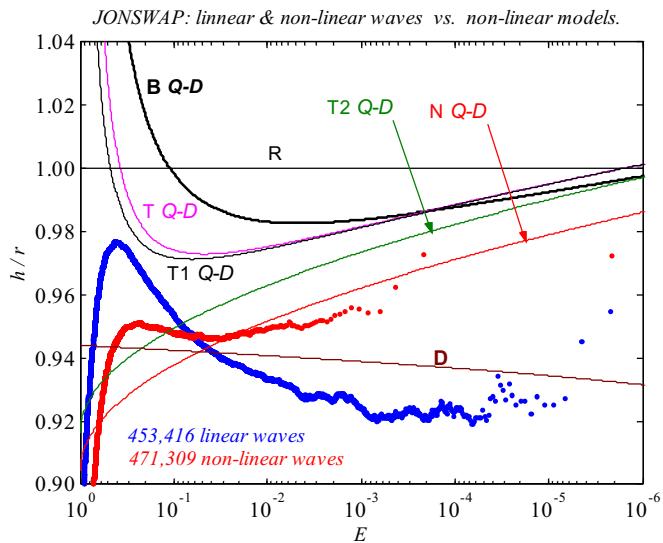


Fig. 4. Linear and nonlinear wave heights vs. theoretical models.

tendency is predicted by all models except D. In this case, N Q-D seems to best mimic the data trend over high waves.

TERN as a whole contains 3,173 waves. Analyzing these in $\frac{1}{2}$ h segments reduces the count of wave heights scaled with the segmental σ to 3,157. These and the corresponding linear model predictions are shown in Fig. 5. It is seen that in contrast with the previous simulations, nonlinearities do not appear to have any significant effect on the observed heights. The data trend is described extremely well by B in general, and by T and T1 over high waves. Most oceanic data should fit this pattern, consistent with past comparisons [9]. None of the nonlinear Q-D type models provide any predictions consistent with TERN. So, they are not included in these comparisons, except for D with the observed average values $\alpha_D = 0.257$ and $\mu_D = 0.119$.

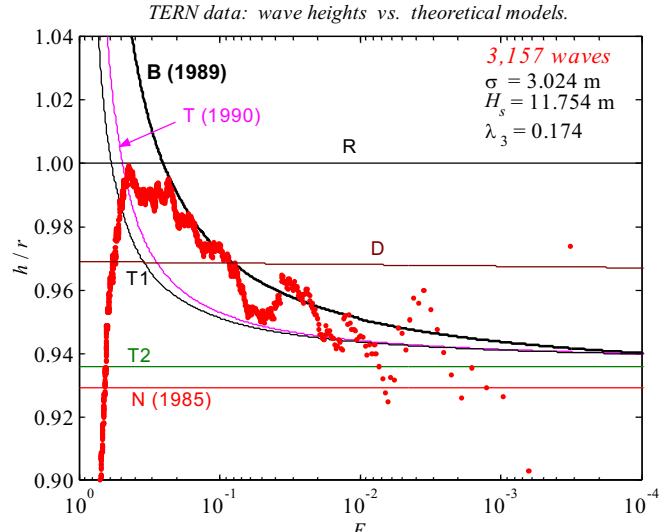


Fig. 5. TERN: observed wave heights vs. linear models.

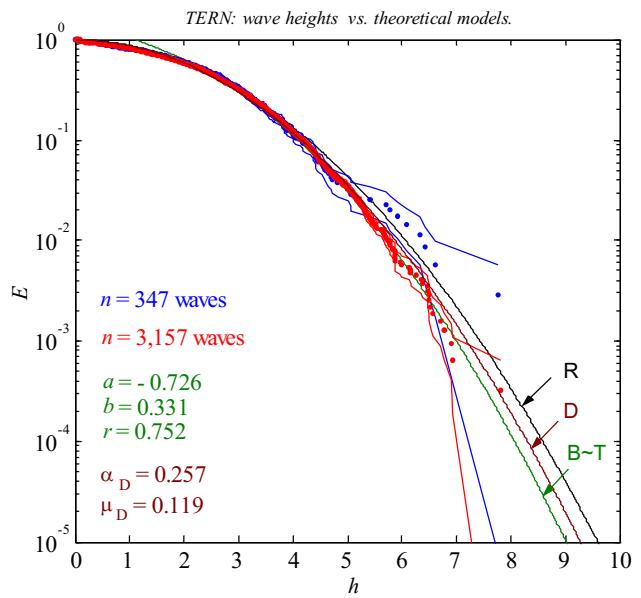


Fig. 6. TERN: effect of sample size n on EDF

To illustrate the effect of the sample size n on the EDF, 3,157 wave heights from TERN are shown again in Fig. 6 in a comparison with 347 wave heights extracted from the third hour of TERN containing the largest wave. Clearly, the shorter hourly data overshoots B, T as well as R over the high wave range where the variability of the EDF estimates is simply too high. And, the apparent trend depicted by the larger wave heights can lead to false conclusions about their relative frequencies of occurrence.

GRAM-CHARLIER APPROXIMATIONS

The joint statistics of the nonlinear wave envelope and phase were considered in [29, 31]. In the present notation, the nonlinear surface elevation and its Hilbert transform are defined by $\eta = \xi \cos \theta$ and $\hat{\eta} = \xi \sin \theta$, with $\xi \equiv$ wave envelope scaled with σ ; and, $\theta \equiv$ wave phase. Formally, $\hat{\eta}$ is the principal value of

$$\hat{\eta} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta(t-\tau)}{\tau} d\tau \quad (18)$$

And, a key point in the formulation is that if η is statistically homogeneous, then

$$\langle \eta \hat{\eta} \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\langle \eta(t) \eta(t-\tau) \rangle}{\tau} d\tau = 0 \quad (19)$$

since the integrand is odd. This allows the joint PDF of η and $\hat{\eta}$ to be expanded in Hermite polynomials more simply as (cf. Eq. 11 in [29])

$$p(\eta, \hat{\eta}) = \frac{1}{2\pi} e^{-\frac{1}{2}(\eta^2 + \hat{\eta}^2)} \left[1 + \sum_{n=0}^3 F_{3-n,n} + \sum_{n=0}^4 G_{4-n,n} \right] \quad (20)$$

where $F_{3-n,n}$ and $G_{4-n,n}$ comprise Hermite polynomials of various orders and the normalized joint cumulants of η and $\hat{\eta}$ to third and fourth order, respectively. The joint PDF of ξ and θ follows from a routine change of variables, and integrating out θ yields the marginal PDF and EDF of ξ as

$$p(\xi) = \xi e^{-\frac{1}{2}\xi^2} [1 + \Lambda (\xi^4 - 8\xi^2 + 8)] \quad (21)$$

$$E(\xi) = e^{-\frac{1}{2}\xi^2} [1 + \Lambda \xi^2 (\xi^2 - 4)] \quad (22)$$

where

$$\Lambda = \frac{1}{64} (\lambda_{40} + 2\lambda_{22} + \lambda_{04}) \quad (23)$$

and

$$\lambda_{40} = \langle \eta^4 \rangle - 3 \quad (24a)$$

$$\lambda_{22} = \langle \eta^2 \hat{\eta}^2 \rangle - 1 \quad (24b)$$

$$\lambda_{04} = \langle \hat{\eta}^4 \rangle - 3 \quad (24c)$$

are the fourth-order normalized cumulants that remain. All of $F_{3-n,n}$ containing the third-order cumulants λ_{nm} for which $n+m=3$, and the terms that contain the fourth-order λ_{31} and λ_{31} in $G_{4-n,n}$ integrate out.

If the definition of wave heights is changed to $h = 2\xi$ as suggested previously in [23], the corresponding PDF and EDF are given, respectively, by

$$p(h) = \frac{h}{4} e^{-\frac{1}{8}h^2} [1 + \frac{\Lambda}{16} (h^4 - 32h^2 + 128)] \quad (25)$$

$$E(h) = e^{-\frac{1}{8}h^2} [1 + \frac{\Lambda}{16} h^2 (h^2 - 16)] \quad (26)$$

In theory, these results are quite general in the sense that they are not conditioned on any restrictive assumptions except for statistical homogeneity. But, the wave-height definition can differ significantly from the crest-to-trough definition as it ignores the variation of the wave envelope in the time interval between a wave crest and the following wave trough. Formally, the difference between the two definitions is of first-order in spectral bandwidth, and that can be significant indeed for relatively wide-band waves typically observed under oceanic conditions.

For second-order narrow-band waves, the joint cumulants will admit various approximations. For example, Tayfun and Lo [29] express all three cumulants in Eq. (23) to second-order in wave steepness and to first-order in spectral bandwidth. The wave envelope is also scaled with $\sigma \sqrt{2}$. If allowance is made for these, it can be verified that the final result given in Eq. (48) of [29] is in fact the same as Eq. (21).

If the spectral bandwidth approaches zero, then $\lambda_{04} = \lambda_{40}$ and $3\lambda_{22} = \lambda_{40}$ correct to second-order in wave steepness [29]. Under these conditions, $h = 2\xi$ approximates the crest-to-trough wave height, and Eq. (26) becomes

$$E(h) = e^{-\frac{1}{8}h^2} \left[1 + \frac{1}{384} \lambda_{40} h^2 (h^2 - 16) \right] \quad (27)$$

Apparently, Mori and Yasuda [17] also draw on the Tayfun and Lo [26] formulation. In what corresponds to Eq. (20) here, some of their conclusions on the nature of the third and fourth-order cumulants disagree with the results in [29]. However, these turn out to be inconsequential as they all integrate out. And, the more recent Eq. (46) derived in Mori and Janssen [18] is identical to the preceding expression, referred to as a modified Edgeworth-Rayleigh (MER) distribution. MER compares quite favorably with the distribution of a relatively small population of about 350-450 waves generated in a 2D flume (see e.g. Fig. 1 in [18]).

To explore the Gram-Charlier type approximations, the distribution of crest-to-trough wave heights gathered from the Draupner platform in the central North Sea is compared in Fig. 7 with MER, R and also with Eq. (26), tagged as G-C. The same data is also plotted in the Forristall style in Fig. 8, which also includes the second-order Q-D forms of T1, T2, N, and also D. Clearly, the observed population size is rather small, comprising just 499 wave heights. These were scaled with the 20-min segmental rms values. The latter differ by less than 5% from the overall average $\sigma \approx 2.922$ m, suggesting that the sea state was reasonably steady during these measurements. The G-C prediction from Eq. (26) uses $\Lambda \approx 0.005$ based on the overall average values of the 20-min segmental fourth-order cumulants. Not surprisingly, these cumulants do not relate to one another in the manner they are expected to do in narrow-band seas. The Draupner data is not narrow-band. One of the five 20-min time series contains the much-celebrated New-Year-Wave, the largest h value plotted in the figure. The MER distribution from Eq. (27) was plotted using the overall average value $\lambda_{40} = 0.272$ of all five 20-min data segments. Evidently, Fig. 8 shows that the larger wave heights follow a trend between MER as an upper bound and N Q-D as an approximate lower bound. The sampling properties of EDFs indicate that definitive conclusions about the nature and significance of the observed deviations from the conventional models would require much larger sample populations, and more systematic and pronounced discrepancies than those displayed by the Draupner data. And, such discrepancies should occur at higher probability levels, viz. for $h \geq 4$, for establishing the relative validity of the G-C type distributions, as in Fig. 1 of Mori and Janssen [18] and in numerous results of Onorato *et al.* [20]. Nonetheless, the results in Figs. 7 and 8 possibly warrant including MER and G-C type distributions in future analyses and comparisons of oceanic data.

EXPECTED SHAPE OF LARGE WAVES

The expected shape of large linear waves has previously been explored by Lindgren [15, 16], Boccotti [2], and Phillips *et al.* [21]. The extensions to second-order waves are more recent and include among others the closed-form formulations of Fedele and Arena (F-A) [7] and Jensen [13, 14]. The F-A model is exact for second-order waves, and Jensen's model relies on

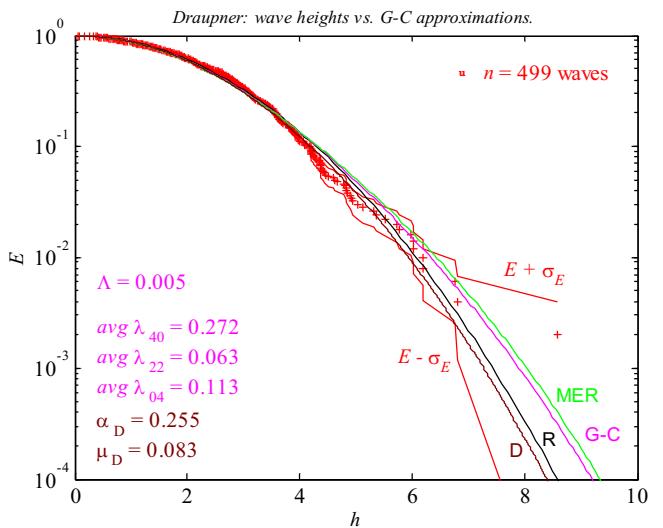


Fig. 7. Draupner data vs. MER, G-C and R models.

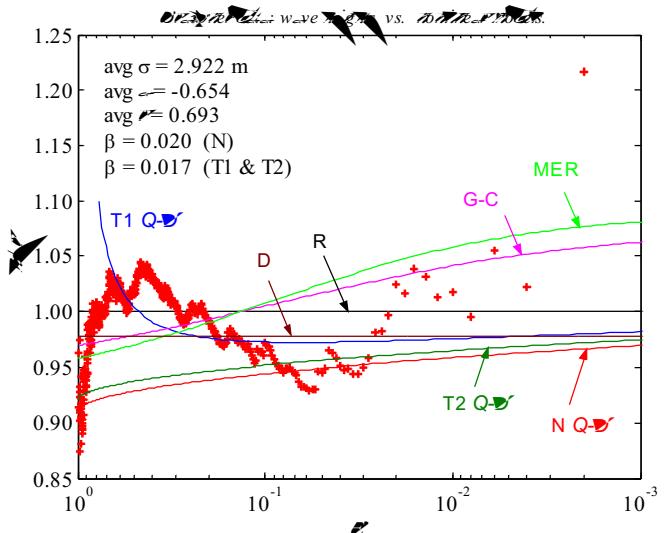


Fig. 8. Draupner data vs. various nonlinear models.

Gram-Charlier type expansions. In F-A, the conditional mean of second-order $\eta(t_0 + \tau)$, given that $\eta(t_0) = \xi \gg 1$ and coincident with a wave crest, is described by Eq. (3). Some applications of this result are in [7] and OMAE2006-92012.

One of Jensen's [13] G-C type representations for the mean shape of large waves has the form

$$\langle \eta(t_0 + \tau) | \eta(t_0) = \xi \rangle = \xi \rho + \frac{1}{2} (\xi^2 - 1) (\lambda_{201} - \rho \lambda_{300}) \quad (29)$$

where $\lambda_{201} = \langle \eta^2(t_0) \eta(t_0 + \tau) \rangle$ and $\lambda_{300} = \lambda_3$. Jensen [13] indicates that the difference between the preceding approximation and his more extensive expression involving additional third-order moments diminishes for large ξ . As in the F-A model of Eq. (3), the statistics ρ , λ_3 and λ_{201} of the model above, say J for brevity, can be derived directly from the surface time series or from the associated spectrum. The latter approach yields more accurate results, but it is also rather impractical as it requires evaluating multiple integrals involving the surface spectrum, as in F-A.

As a simpler alternative, consider the second-order narrow-band (N-B) model [27, 32], viz.

$$\eta = \eta_1 + \frac{1}{2} \mu (\eta_1^2 - \hat{\eta}_1^2) \quad (30)$$

where $\mu = \lambda_3 / 3$ and $\hat{\eta}_1$ \equiv Hilbert transform of η_1 . It requires some algebra coupled with the joint statistics of η_1 and $\hat{\eta}_1$ to verify that the conditional mean of $\eta(t_0 + \tau)$, given that $\eta(t_0) = \xi$ and coincident with a wave crest, is given by

$$\begin{aligned} \langle \eta(t_0 + \tau) | \eta(t_0) = \xi \rangle &= \rho \xi_1 \\ &+ \frac{1}{2} \mu (\xi_1^2 - 1) (\rho^2 - \hat{\rho}^2) \end{aligned} \quad (31)$$

where $\hat{\rho}(\tau) \equiv \langle \hat{\eta}(t_0 + \tau) \eta(t_0) \rangle$ and corresponds to the Hilbert transform of ρ . In the actual results, both ρ and $\hat{\rho}$ are defined in terms of η_1 . But, they are equally well defined in terms of η , correct to $O(\mu)$, and this is entirely consistent with the second-order theory. Again, both ρ and $\hat{\rho}$ can be derived from the surface time series directly or from the cosine and sine transforms of the associated spectral estimates. In this case, the latter approach not only yields more accurate estimates, but it is also considerably easier than it can be implemented in the F-A and J models.

Note that setting $\tau = 0$ in Eq. (31) will give

$$\xi = \xi_1 + \frac{1}{2} \mu (\xi_1^2 - 1) \quad (32)$$

Consequently,

$$\xi_1 = \frac{1}{\mu} (-1 + \sqrt{1 + 2\mu\xi + \mu^2}) \quad (33)$$

on the right-hand side of Eq. (31).

For the New-Year-Wave, $\xi = \eta_{\max} = 6.204$, $\mu = 0.137$, and $\xi_1 = 4.733$ from Eq. (33). The actual profile scaled with the corresponding segmental $\sigma = 2.981$ m is plotted in Fig. 9. The same figure also displays the mean N-B shape from Eq. (31) using the spectral estimates of ρ and $\hat{\rho}$, and the corresponding linear Q-D form [2, 15, 16, 21]

$$\langle \eta(t_0 + \tau) | \eta(t_0) = \xi \rangle = \xi \rho(\tau) \quad (34)$$

It is seen that because the preceding expression ignores the nonlinear nature of the actual profile, it does not depict the magnitude and shape of the New-Year-Wave as well as the nonlinear N-B model does. The N-B model approximates the actual profile surprisingly well indeed. This is despite the fact that the New-Year-Wave is a random realization from 3D directional seas whereas the N-B profile is a statistical expectation appropriate to 2D waves. There is a slight horizontal distortion in the actual profile in that the front of the wave is steeper than its rear, quite possibly because of surface stresses. The low sampling rate (~ 2.13 Hz) of the Draupner data as a fixed-point time series suggests that the actual wave is possibly larger than the New-Year-Wave. All aside, Fig. 9 suggests that the New-Year-Wave is likely to be a relatively “rare realization of a typical population” as opposed to being “a typical realization of a rare population” [11].

As an additional comparison, Fig. 10 shows the New-Year-Wave together with J from Eq. (29) and N-B from Eq. (31), using in this case λ_3 , λ_{201} , ρ and $\hat{\rho}$ estimated from the 20-min time series. Contrasting Figs. 9 and 10, it is seen that the N-B profile in the latter figure does not compare with the observed wave as favorably as the spectrum-based prediction in Fig. 9. The J prediction does quite well, particularly around the large wave crest, but it becomes more oscillatory than the observed profile away from the wave crest. A more comprehensive comparison of J given in [14] uses Jensen’s full formulation and spectral expressions, and thus compares with the New-Year-Wave noticeably better than the present J prediction does.

CONCLUDING REMARKS

The 2D linear simulations carried out here suggest that the models of Boccotti [1] and Tayfun [25] yield similar predictions. Both models, in particular, Boccotti’s describe the data trends over relatively large waves quite accurately. The widely popular Naess [16] model under predicts the linear wave heights somewhat, as in previous comparisons [28]. Both approximations (T1 & T2) of the Tayfun model are just as simple as the Naess model, and they also describe the linear wave heights slightly and consistently better than the Naess model.

The 2D nonlinear simulations for narrow-band long-crested seas lead to a re-distribution of wave heights biased slightly toward the higher waves. For this extreme and possibly rather

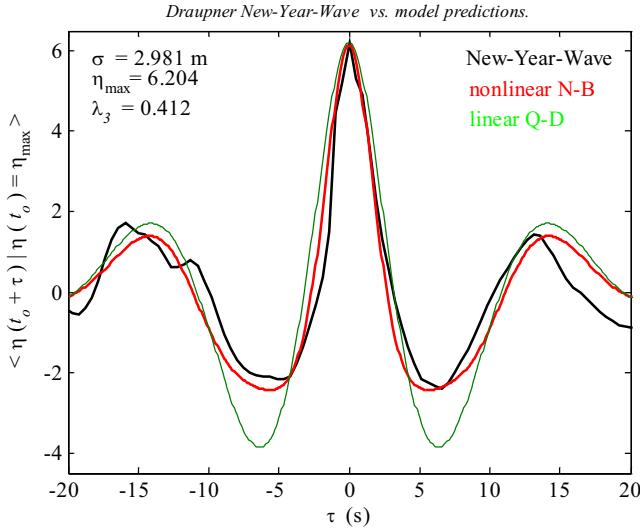


Fig. 9. New-Year-Wave vs. mean linear Q-D and nonlinear N-B profiles derived from spectral estimates.

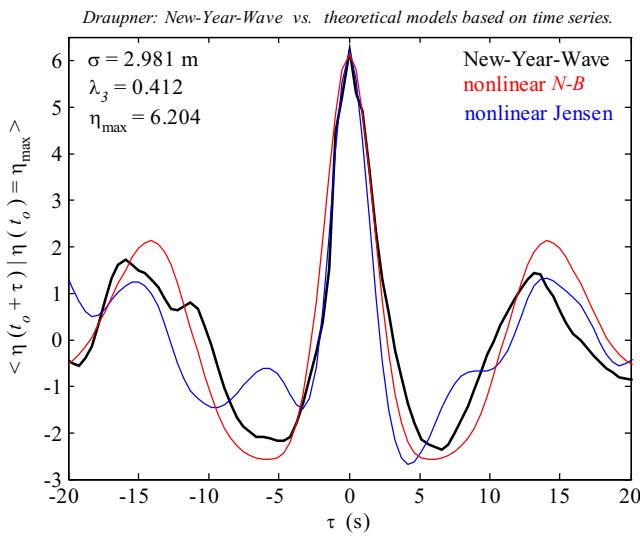


Fig. 10. New-Year-Wave vs. mean nonlinear J and N-B profiles derived from surface time series.

unrealistic case, the non-linear models considered describe the data trend fairly satisfactorily, except for the Dawson [5] model. Of all the nonlinear extensions considered, the Naess model seems to best describe the simulated wave heights. However, the analysis of TERN data does not indicate any similar nonlinear effects, and the observed wave heights are described extremely well by Boccotti's model [1]. This in essence confirms previous comparisons and is likely to be a typical result under oceanic conditions [9].

Janssen [12] and Onorato *et al.* [20] demonstrate that the heights of 2D narrow-band waves generated in wave flumes can be amplified rather noticeably due to third-order free-wave

interactions. In particular, Onorato *et al.* [20] show that the conventional linear and second-order probability laws do not explain the statistics of wave heights and crests in the presence of such interactions. And, Mori and Janssen [18] show that wave heights under the same conditions are better described by a modified Edgeworth-Rayleigh form of the Gram-Charlier series dependent on the kurtosis of surface elevations. However, whether if the same conclusions apply to oceanic waves remains essentially unknown [20]. The results on directional simulations in Socquet-Juglard [23] and related other studies indicate that this may not be the case. The present analyses of the TERN and Draupner data lead to a similar conclusion, and suggest that the discrepancies observed under oceanic conditions are most likely due to the sampling variability of estimates derived from relatively short samples. Further, the present results from the TERN and Draupner data do not provide any clear evidence for the efficacy of the Gram-Charlier type expansions. In fact, some recent comparisons in Cherneva *et al.* [4] clearly indicate that such approximations do rather poorly in representing wave height characteristics. Nonetheless, further comparisons with larger populations of oceanic measurements would be needed to resolve this issue unequivocally.

In the past, fixed-point measurements of the sea surface served well in expanding our knowledge about oceanic waves, but they may no longer be adequate in providing sufficiently accurate data on the extremes of the surface. In theory, the probability that a surface time series gathered at a fixed point includes the true surface maxima and minima is essentially nil, except when waves are atypically long-crested. Crest and trough heights derived from such time series can and probably often do seriously underestimate the actual values. This is a major point of concern in assessing the validity of results derived from the TERN and Draupner data, and possibly those in Cherneva *et al.* [4]. It does not arise in the results of Mori and Janssen [18] and Onorato *et al.* [20]. And, the question as to how well a particular model does in describing the statistics of extremely large wave heights or crests can not be fully resolved unless the sea surface is measured in both time and space.

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