

NOVEL NUMERICAL TECHNIQUES FOR  
PROBLEMS IN ENGINEERING SCIENCE

A Dissertation Presented

by

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to

The Faculty of the Graduate College

of

The University of Vermont

In Partial Fulfillment of the Requirements  
for the Degree of Doctor of Philosophy  
Specializing in Civil and Environmental Engineering

February, 2005

Accepted by the Faculty of the Graduate College, The University of Vermont, in partial fulfillment of the requirements for the Degree of Doctor of Philosophy, Specializing in Civil and Environmental Engineering

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# Abstract

In this thesis novel numerical techniques are proposed for application to different problems in engineering science. Three categories of numerical techniques are investigated: collocation methods, finite element methods (FEM) and boundary element methods (BEM).

In the context of collocation methods a new numerical technique called LOCOM (LOCALized COLlocation Method) has been proposed. This method is able to reduce the degrees of freedom of the classical Hermite collocation to one single degree for each collocation node, still maintaining higher order accuracy. This new methodology has been applied to an existing Hermite Collocation Fortran code that solves multiphase flow problems.

In the context of the Finite Element Method, a special form of the Petrov-Galerkin method has been formulated for the sub-grid stabilization of advection-diffusion partial differential equations on triangular meshes. This new method is able to damp out the spurious oscillations occurring near a sharp front when the standard finite element method is applied. An adjoint FEM has been developed in the context of fluorescence tomography and a Galerkin technique has been formulated to investigate the hydrodynamic stability of pulsatile Poiseuille flow in a pipe.

Finally, a 3D boundary element method is presented for the numerical solution of general coupled elliptic differential equations. This methodology has application in some areas of optical tomography where small heterogeneities immersed in a homogenous domain need to be detected.

# Citations

Material from this dissertation has been published in the following form:

Fedele F., Hitt D., Prabhu R.D. Revisiting the stability of Pulsatile pipe flow. European Journal of Mechanics - B/Fluids 2004, in press

Fedele F., Melissa Mckay, G. F. Pinder and Guarnaccia J. A single-degree of freedom Hermite Collocation for multi-phase flow and transport in porous media. Inter. Journal Numerical methods in fluids 44:1337-1354, 2004

Fedele F., Laible J. P. & Eppstein M. Coupled complex adjoint sensitivities for frequency-domain fluorescence tomography: theory and vectorized implementation ” Journal of Computational physics Vol. 187, Issue 2, pp. 597-619, 2002

Fedele F., Laible J. P., Pinder G. F. Localized-Adjoint-Finite-Element-Method for Sub-Grid Stabilization of Convection-dominated Transport on a Triangular Mesh. XIV International Conference on Computational Methods in Water Resources June 23-28, 2002 Delft University of Technology The Netherlands

Material from this dissertation has been submitted for publication to Journal of Computational Physics on 07/01/04 in the following form:

Fedele F., Laible J. P., A. Godavarty, E. M. Sevick-Muraka, Eppstein M. Fluorescence Photon migration by the Boundary Element Method.

## Acknowledgement

First of all I want to thank my advisor, Jeffrey Laible, for introducing and assisting me in the fascinating world of Finite Element Methods and George Pinder who trusted my capabilities and gave me a chance to study here in the United States. I also thank Tullio Tucciarelli for giving me the opportunity to study abroad after I finished my master in Italy and meet George Pinder as a visiting student here at UVM. I also want to thank all the people I worked with during these years: Darren Hitt with whom I revisited the world of fluid mechanics, R. D. Prabhu with whom I discovered the beautiful world of fluid turbulence, Jianke Yang with whom I discovered the fascinating world of nonlinear equations and Igor Najfeld whose advice and suggestions have been extremely helpful to me. I'm also grateful to my co-advisor Margaret Eppstein for her support and help in a difficult moment I had during my studies. I also want to name some of my friends and colleagues in Burlington: Adam, Ceyhun, Edward, Metin and Simon. They have been very close to me. Grazie soprattutto a mia mamma Maria Anna, mio nonno Rocco, mio fratello Marco e mia sorella Giovanna per esser stati sempre vicino a me anche a grande distanza da casa. In ricordo di mio padre, dedico questa tesi a lui. Se fosse ancora in vita sarebbe molto orgoglioso di suo figlio. Il conseguimento del dottorato di ricerca e' la piu' grande soddisfazione che avrei potuto dargli dopo tutte le umiliazioni subite prima che morisse.

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