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Successive wave crests in Gaussian seas

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Abstract

In this paper, following the theory of quasi-determinism of Boccotti [Boccotti P. Wave mechanics for ocean engineering. Oxford: Elsevier Science.], the necessary and sufficient conditions, for the occurrence of two successive wave crests of large heights in a gaussian sea, are given. It is proven that the first two-peaks part of the autocovariance function $\psi(T)$ describes the structure of two successive-wave patterns. As a corollary, it is shown that the tail probability of the joint distribution of two successive wave crests is given by a bivariate Weibull distribution. The Weibull parameter is equal to $\psi_2^* = \psi(T_2^*)/\psi(0)$. Here, T_2^* is the abscissa of the second absolute maximum of the autocovariance function $\psi(T)$. The analytical results are in agreement with Monte Carlo simulations. Finally, as an application, the maximum expected wave crest pressure in an undisturbed deep water waves is evaluated by taking into account the stochastic dependence of successive wave crests.

Keywords: Successive wave crests; Gaussian sea; Quasi-determinism; Bivariate Weibull; Conditional probability; Maximum expected wave crest

1. Introduction

The theory of quasi-determinism for the mechanics of linear wave groups was derived by Boccotti in the eighties, with two formulations. The first one [1–3] enables us to predict what happens when a very high crest occurs at a fixed time and location (see also [4–10]); the second one [11–14] gives the mechanics of the wave group when a very large crest-to-trough height occurs. The theory, which is exact to the first order in a Stokes expansion (Gaussian sea), is valid for any boundary condition (for example either for waves in an undisturbed field or in reflection). The theory was then verified in the nineties with some small-scale field experiments [15,16], both for waves in an undisturbed field and for waves interacting with structures. Boccotti [14] then proposed a complete review of the theory. He showed that both the highest wave height or the highest wave crest are different occurrences of the space-time evolution of a well defined wave group. Thus, the two formulations are congruent to each other. An alternative approach for the derivation of the quasi-determinism theory was proposed by Phillips et al. [17,18], who also obtained a field verification off the Atlantic coast of the USA. The first formulation of the theory (derived only for the time domain) was also given by Tromans et al. [19], who renamed the theory as 'New Wave'.

In this paper the theory of sea states is summarized first, then the second formulation theory of quasi-determinism is revisited, in order to emphasize the key steps of the proof of Boccotti. In his theory, setting t_0 as an arbitrary time instant, H the wave height and T^* as the abscissa of the absolute minimum of the autocovariance function $\psi(T)$, Boccotti showed that as $\alpha = H/\sigma \to \infty$ the condition

$$\eta(t_0) = \frac{H}{2} \quad \text{and} \quad \eta(t_0 + T^*) = -\frac{H}{2} \tag{1}$$

becomes necessary and sufficient for the occurrence of a wave of height H (being σ the standard deviation of the surface displacement η). As a corollary, in the limit of $\alpha \to \infty$, Boccotti derived that the probability of exceedance of the wave height follows the Weibull distribution (see also [20,21])

$$\Pr(H > \alpha) = \exp\left[-\frac{\alpha^2}{4(1+\psi^*)}\right]. \tag{2}$$

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Here, ψ^* is the narrow-bandedness parameter defined as the absolute value of the quotient between the first absolute minimum and the absolute maximum of $\psi(T)$.

Following Boccotti ([11,14]), the necessary and sufficient conditions for the occurrence of two successive wave crests of very large given height are provided. It is also proven that the first two-peaks part of the autocovariance function $\psi(T)$ describes the structure of two successive-wave patterns. As a corollary, it is shown that the tail probability of the joint distribution of two successive wave crests is given by a bivariate Weibull distribution. The analytical results are then validated by Monte Carlo simulations. Finally, as an application, the maximum expected wave crest pressure in an undisturbed deep water waves is computed.

2. The theory of sea states

According to the theory of sea states, to the first order in a Stokes expansion, a time series of surface displacements $\eta(t)$, recorded at a fixed point at sea, is a realization of the stationary ergodic stochastic Gaussian process

$$\eta(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t + \varepsilon_i). \tag{3}$$

Here, it is assumed that frequencies ω_i are different from each other, the number N is infinitely large and the phase angles ε_i, uniformly distributed in $[0, 2\pi]$, are stochastically independent of each other. Furthermore, all the amplitudes a_i satisfy the frequency spectrum $S(\omega)$ defined as

$$S(\omega)\Delta\omega = \sum_i \frac{a_i^2}{2} \quad \omega_i \in \left(\omega - \frac{\Delta\omega}{2}, \omega + \frac{\Delta\omega}{2}\right). \tag{4}$$

The jth order moment of the spectrum is $m_j = \int_0^\infty \omega^j S(\omega)\mathrm{d}\omega$. In particular $m_0 = \sigma^2$, where σ is the standard deviation of $\eta(t)$. The autocovariance function $\psi(T)$ can be evaluated as

$$\psi(T) = \int_0^\infty S(\omega)\cos(\omega T)\mathrm{d}\omega.$$

In the context of ocean waves the JONSWAP spectrum [22] is adopted in the following form

$$S(\omega) = Ag^2\omega_p^{-5}\left(\frac{\omega}{\omega_p}\right)^{-5} \exp\left[-\frac{5}{4}\left(\frac{\omega}{\omega_p}\right)^{-4}\right] \cdot \exp\left\{\ln\gamma \exp\left[-\frac{(\omega-\omega_p)^2}{2\chi_2^2\omega_p^2}\right]\right\}. \tag{5}$$

Here, ω_p is the peak frequency, A is the Phillips parameter, γ is the enhancement coefficient. For typical wind waves one can assume $\gamma = 3.3$ and $\chi_2 = 0.08$. For $\gamma = 1$ and $A = 0.0081$ the Pierson–Moskowitz spectrum is recovered. Note that other wave processes, to the first order in a Stokes expansion, can be expressed as in Eq. (3) with appropriate choice of the spectral coefficients $\{a_i\}_{i=1,N}$. Note that the moments of the JONSWAP spectrum exist as far as m_3 implying only the existence of the first-order derivative of the stochastic process $\eta(t)$. By cutting off the high-frequency tail of the JONSWAP spectrum, one can define a new spectrum over a compact support and all the moments then exist implying the existence of all the higher derivatives of $\psi(T)$ and consequently the (m.s. stochastic) differentiability of $\eta(t)$ to any order. As pointed out by Boccotti [see [14], p. 294] 'the high frequency term does not alter the crest elevation, nor the trough depth, nor the time interval between the crest and trough, nor the wave period. It simply ruffles the wave surface with a lot of very small ripples'. In the applications, the JONSWAP spectrum $S(\omega)$ can be considered in the frequency range $\omega \in [0, 6\omega_p]$ when the interest is in the analysis of the wave crest, wave trough or wave height.

3. The theory of quasi-determinism

Let us consider the surface displacement $\eta(t)$ at any fixed point (x_0, y_0) in a Gaussian wave field. Setting t_0 as an arbitrary time instant, H as the wave height and T^* as the abscissa of the absolute minimum of $\psi(T)$, Boccotti [11,13] showed that the condition (1) is necessary and sufficient for the occurrence of a wave of height H as $\alpha = H/\sigma \to \infty$. The condition (1) is sufficient because as $\alpha \to \infty$ the conditional p.d.f.

$$p\left[\eta(t_0+T) = u / \eta(t_0) = \frac{H}{2}, \eta(t_0+T^*) = -\frac{H}{2}\right] \tag{6}$$

tends to a delta function $\delta[u - \bar{\eta}(t_0+T)]$ centered at

$$\bar{\eta}(t_0+T) = \frac{H}{2}\frac{\psi(T) - \psi(T-T^*)}{\psi(0) - \psi(T^*)}. \tag{7}$$

This implies that as $\alpha \to \infty$, given the condition (1), with probability approaching one, the surface displacement $\eta(t_0+T)$ tends to the deterministic form $\bar{\eta}(t_0+T)$. This is a wave profile with wave height H, having a crest of amplitude $H/2$ at $T=0$ and a trough of amplitude $H/2$ at $T=T^*$.

In order to show that condition (1) is also a necessary condition, Boccotti derived the analytical expression for the expected number per unit time $\mathrm{EX}(\alpha, \tau, \xi)$ of local maxima of the surface displacement $\eta(t)$ with amplitude $\xi\alpha$ which are followed by a local minimum with amplitude $(\xi - 1)\alpha$ after a time lag τ. He showed that as $\alpha \to \infty$ in the domain (τ, ξ) there exists an $O(\alpha^{-1})$ infinitesimal neighborhood ($\delta\tau$, $\delta\xi$) of (T^*, 1/2) such that

$$\mathrm{EX}_{s.w.}(\alpha,\tau,\xi) = \begin{cases} \mathrm{EX}\left(\alpha, T^*, \frac{1}{2}\right)\exp\left[-\frac{1}{8}(K_\tau^*\delta\tau^2 + K_\xi^*\delta\xi^2)\alpha^2\right] & \\ 0 & \text{elsewhere} \end{cases}.$$

Here, K_τ^* and K_ξ^* are constants and $\mathrm{EX}_{\mathrm{s.w.}}(\alpha, \tau, \xi)$ is the expected number per unit time of local maxima of the surface displacement $\eta(t)$ with amplitude $\xi\alpha$ which are followed by a local minimum with amplitude $(\xi-1)\alpha$ after a time lag τ, *where the local maximum and the local minimum must be the crest and the trough of the same wave, respectively*, (the subscript *s.w.* stands for *same wave*). Thus, a local maximum of dimensionless amplitude $\alpha/2+\delta\xi$ followed by a local minimum of dimensionless amplitude $\alpha/2-\delta\xi$ after a time lag $T^*+\delta\tau$ has almost the same maximal expectation as a local maximum with amplitude $\alpha/2$ followed by a local minimum of amplitude $\alpha/2$ lagged in time by T^* in the limit of $\alpha\to\infty$. But a local maximum and a local minimum of amplitudes $\alpha/2$ lagged in time by T^* are also the crest and trough of a wave because condition (1) is sufficient. Hence, condition (1) is also necessary in the limit of $\alpha\to\infty$.

As a corollary, Boccotti showed that the wave height distribution $p(\alpha)$ (see also [20,21]) admits the following asymptotic expression

$$p(\alpha) = \frac{\int_0^\infty\int_0^1 \mathrm{EX}_{\mathrm{s.w.}}(\alpha,\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi}{\mathrm{EX}_+} = \frac{\alpha}{2(1+\psi^*)}\exp\left[-\frac{\alpha^2}{4(1+\psi^*)}\right]$$

as $\alpha\to\infty$.

Here, $\psi^* = -\psi(T^*)/\psi(0)$ is the narrow-bandedness parameter and

$$\mathrm{EX}_+ = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}} \tag{9}$$

is the expected number per unit time of zero up-crossing of the surface displacement. For narrow-band spectra $\psi^*\to 1$, whereas broad-band spectra are characterized by $\psi^*\ll 1$.

4. The occurrence of two successive wave crests of very large heights

4.1. Sufficient conditions

In the following, the theory of quasi-determinism of Boccotti is extended to study the occurrence of two very large successive wave crests. Consider the probability density function of the surface displacement $\eta(t)$, at any fixed point (x_0, y_0) in a Gaussian sea, given the conditions

$$\eta(t_0) = h_1 \quad\text{and}\quad \eta(t_0+T_2^*) = h_2. \tag{10}$$

Here, t_0 is an arbitrary time instant, h_1 and h_2 are crest amplitudes and T_2^* is the abscissa of the second absolute maximum of the autocovariance function $\psi(T)$ (see Fig. 1). The p.d.f. of $\eta(t)$ at time t_0+T, given conditions (10) is Gaussian, i.e.

$$p[\eta(t_0+T) = u/\eta(t_0) = h_1, \eta(t_0+T_2^*) = h_2] = \frac{1}{\sqrt{2\pi\sigma_c^2}}\exp\left\{-\frac{[u-\eta_c(t_0+T)]^2}{2\sigma_c^2}\right\}$$

where the conditional mean $\eta_c(t_0+T)$ is given by

$$\eta_c(t_0+T) = C_1\psi(T) + C_2\psi(T-T_2^*) \tag{11}$$

and the coefficients C_1 and C_2 are given by

$$C_1 = \frac{h_1\psi(0)-h_2\psi(T_2^*)}{\psi^2(0)-\psi^2(T_2^*)}, \qquad C_2 = \frac{h_2\psi(0)-h_1\psi(T_2^*)}{\psi^2(0)-\psi^2(T_2^*)}. \tag{12}$$

The conditional variance σ_c^2 admits the following expression

$$\frac{\sigma_c^2}{\sigma^2} = 1 - \frac{\psi^2(T)+\psi^2(T-T_2^*)-2\psi(T)\psi(T-T_2^*)\frac{\psi(T_2^*)}{\psi(0)}}{\psi^2(0)-\psi^2(T_2^*)}. \tag{13}$$

It follows that $\sigma_c<\sigma$ since $\psi(T_2^*)/\psi(0)$ is smaller than unity by definition. This implies that, in the limit of $h_1/\sigma\to\infty$ and $h_2/\sigma\to\infty$ the ratio $\sigma_c/\eta_c(t_0+T)$ approaches zero, since $\eta_c(t_0+T)\to\infty$ and σ_c is bounded by the unconditional standard deviation σ. Thus, all the realizations of the Gaussian sea satisfying conditions (10), with probability approaching one, tend to the deterministic profile $\eta_c(t_0+T)$

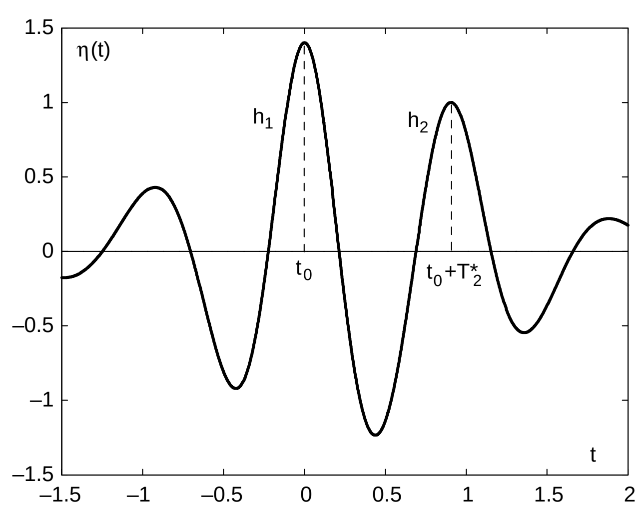

Fig. 1. Two successive wave crests lagged in time by T_2^*.

for very large crest heights, i.e.

$$p[\eta(t_0+T)=u/\eta(t_0)=h_1,\eta(t_0+T_2^*)=h_2] \rightarrow \delta[u-\eta_c(t_0+T)] \quad \text{as} \quad \frac{h_1}{\sigma} \quad \text{and} \quad \frac{h_2}{\sigma}\rightarrow\infty.$$

The conditional mean $\eta_c(t_0+T)$, [see Eq. (11)] represents a wave structure of two successive wave crests lagged in time by T_2^*, if certain constraints are given. Note that $\eta_c(t_0+T)$ is a linear combination of the autocovariance $\psi(T)$ and the shifted autocovariance $y(T-T_2^*)$. Since, $T=0$ and $T=T_2^*$ are the abscissa of the first absolute maximum and second absolute maximum of $\psi(T)$ respectively, this implies that $\eta_c(t_0+T)$ attains two local maxima at $T=0$ and $T=T_2^*$, if both the second order derivatives at these abscissas are less than zero, i.e.

$$\ddot{\eta}_c(0)<0 \quad \text{and} \quad \ddot{\eta}_c(T_2^*)<0. \tag{14}$$

Some algebra yields

$$\ddot{\eta}_c(0)=a(-\beta_0+s\beta_1) \qquad \ddot{\eta}_c(T_2^*)=a(-\beta_1+s\beta_0) \tag{15}$$

where $\beta_0=h_1/\sigma$ and $\beta_0=h_2/\sigma$ and (the dot denotes time derivative)

$$a=\frac{1+\psi(T_2^*)\ddot{\psi}(T_2^*)}{1-\psi^2(T_2^*)} \qquad s=\frac{\psi(T_2^*)+\ddot{\psi}(T_2^*)}{1+\psi(T_2^*)\ddot{\psi}(T_2^*)}.$$

Since, a is always greater or equal to zero, the condition (14) is fulfilled if

$$\begin{cases}\beta_0,\beta_1\in\mathbb{R}_+^2 & \text{if } s\le 0\\ \beta_0,\beta_1\in\Omega(s) & \text{if } s>0\end{cases}. \tag{16}$$

Here, $\Omega(s)$ is the open sectorial region of $\mathbb{R}_+^2$ with aperture angle $\theta=\pi/2-2\tan^{-1}(s)$, that is

$$\Omega(s)=\left\{(\beta_0,\beta_1)\in\mathbb{R}_+^2:\beta_0\ge 0,\beta_1\ge 0,s<\frac{\beta_1}{\beta_0}<\frac{1}{s}\right\}.$$

Typical JONSWAP spectrum satisfies the condition $s>0$ with $s\in[0.14,0.16]$. As the spectrum gets narrow the sector $\Omega(s)$ tends to cover all $\mathbb{R}_+^2$, i.e. $\theta\rightarrow\pi/2$, because s approaches zero in the narrow-band limit. Moreover, since it is assumed that the autocovariance function $\psi(T)$ attains only one minimum at $T=T^*$ in the open interval $(0,T_2^*)$ (see Boccotti, [13,14]), the two local maxima of the wave profile $\eta_c(t_0+T)$ are also two consecutive wave crests (see Fig. 1). Hence, as $\beta_0\rightarrow\infty$ and $\beta_1\rightarrow\infty$, conditions (10) are sufficient for the occurrence of two successive wave crests of very large height within the limits of constraint (16).

4.2. The Conditions (10) are necessary for the occurrence of two large successive wave crests

In the following, the notations ψ_T, η_T are adopted to indicate respectively the autocovariance $\psi(T)$ and the surface displacement $\eta(T)$. Without losing generality, the time scale $m_0/\sqrt{m_2}$ and the length scale $\sigma=\sqrt{m_0}$ are used to non-dimensionalize Eq. (3) such that the zeroth and the second order moment of the spectrum are equal to one, i.e. $m_0=1$ and $m_2=1$. It follows that $\psi_0=1$ and $\ddot{\psi}_0=-1$. Consider the expected number per unit time

$$\mathrm{EX_c}(\beta_0,\beta_1,\tau)\mathrm{d}\beta_0\mathrm{d}\beta_1\mathrm{d}\tau \tag{17}$$

of local maxima of the surface displacement $\eta(t)$ (at a fixed location in space) whose elevation is between β_0 and $\beta_0+\mathrm{d}\beta_0$, and which are followed by a local maximum with an elevation between β_1 and $\beta_1+\mathrm{d}\beta_1$ after a time lag between τ and $\tau+\mathrm{d}\tau$. Following the general approach introduced by Rice (see [14], pp. 159–162), $\mathrm{EX_c}(\beta_0,\beta_1,\tau)$ can be expressed as

$$\mathrm{EX_c}(\beta_0,\beta_1,\tau)=\int_{-\infty}^{0}\int_{-\infty}^{0}|z_1z_2|p[\eta_0=\beta_0,\dot{\eta}_0=0,\ddot{\eta}_0=z_1,\eta_\tau=\beta_1,\dot{\eta}_\tau=0,\ddot{\eta}_\tau=z_2]\mathrm{d}z_1\mathrm{d}z_2. \tag{18}$$

Here, $p[\eta_0,\dot{\eta}_0,\ddot{\eta}_0,\eta_\tau,\dot{\eta}_\tau,\ddot{\eta}_\tau]$ is a Gaussian joint probability density function. Eq. (18) is rewritten in the form

$$\mathrm{EX_c}(\beta_0,\beta_1,\tau)=p[\eta_0=\beta_0,\dot{\eta}_0=0,\dot{\eta}_\tau=0,\eta_\tau=\beta_1]\int_{-\infty}^{0}\int_{-\infty}^{0}|z_1z_2|\cdot p[\ddot{\eta}_0=z_1,\ddot{\eta}_\tau=z_2/\eta_0=\beta_0,\dot{\eta}_0=0,\dot{\eta}_\tau=0,\eta_\tau=\beta_1]\mathrm{d}z_1\mathrm{d}z_2.$$

Since, the conditional mean η_c attains two local maxima at $T=0$ and $T=T_2^*$, in the limit of $\beta_0\rightarrow\infty$ and $\beta_1\rightarrow\infty$ the following holds

$$p[\ddot{\eta}_0=z_1,\ddot{\eta}_\tau=z_2/\eta_0=\beta_0,\dot{\eta}_0=0,\dot{\eta}_\tau=0,\eta_\tau=\beta_1] \rightarrow\delta[z_1-\ddot{\eta}_c(0),z_1-\ddot{\eta}_c(T_2^*)]$$

and this yields the simplification of Eq. (18) as follows

$$\mathrm{EX_c}(\beta_0,\beta_1,\tau)=p[\eta_0=\beta_0,\dot{\eta}_0=0,\dot{\eta}_\tau=0,\eta_\tau=\beta_1]\ddot{\eta}_c(0)\ddot{\eta}_c(T_2^*). \tag{19}$$

Here, if the joint probability $p[\eta_0=\beta_0,\dot{\eta}_0=0,\dot{\eta}_\tau=0,\eta_\tau=\beta_1]$ is Taylor expanded with respect to the variable τ around $\tau=T_2^*$ (see Appendix), this gives

$$\mathrm{EX_c}(\beta_0,\beta_1,\tau)\simeq\frac{1}{(2\pi)^2\sqrt{1-\psi_{T_2^*}^2}}\cdot\exp\left[-\frac{\beta_0^2+\beta_1^2-2\psi_{T_2^*}\beta_0\beta_1}{2(1-\psi_{T_2^*}^2)}-\frac{K^*}{2}\delta\tau^2+o(\delta\tau^2)\right]. \tag{20}$$

Note that the coefficient K^* is greater than zero and tends to infinity as $\beta_0\rightarrow\infty$ and $\beta_1\rightarrow\infty$ (see Appendix). Hence, in the same limit, from Eq. (20) there exists an infinitesimal neighborhood $\delta\Gamma$ of order $O(K_*)$ such that $\forall\,\delta\tau\in\delta\Gamma$

$$\mathrm{EX_c}(\beta_0,\beta_1,\tau) = \begin{cases} \mathrm{EX_c}(\beta_0,\beta_1,T_2^*)\exp\left(-\frac{1}{2}K^*\delta\tau^2\right) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{as} \quad \beta_0\to\infty \quad \text{and} \quad \beta_1\to\infty. \tag{21}$$

Numerical investigations show that

$$\frac{\mathrm{EX_c}(\beta_0,\beta_1,T_2^*)}{\mathrm{EX_c}(\beta_0,\beta_1,\tau)} > 1 \quad \forall\,\tau \neq T_2^*.$$

Thus, two successive local maxima of dimensionless amplitude β_0 and β_1 respectively, attain the maximal expectation $\mathrm{EX_c}(\beta_0, \beta_1, \tau)$ when the time lag between their occurrence is equal to $\tau = T_2^*$. Moreover, from Eq. (21) a local maxima of a very large amplitude β_0 followed by a local maxima of a very large amplitude β_1 after a time lag $T_2^* + \delta\tau$ or have almost the same maximal expectation as two local maxima with amplitudes equal to β_0 and β_1 respectively, lagged in time by T_2^*. However, two local maxima of large amplitude lagged in time by T_2^* are also two successive crests because the conditions (10) are sufficient. Hence, the conditions (10) are also necessary in the limit of $\beta_0\to\infty$ and $\beta_1\to\infty$.

5. The tail probabilities of two successive wave crest heights

Let us define

$$\mathrm{EX_{s.c.}}(\beta_0,\beta_1,\tau)\mathrm{d}\beta_0\mathrm{d}\beta_1\mathrm{d}\tau \tag{22}$$

as the expected number per unit time of local maxima of the surface displacement $\eta(t)$ (at a fixed location in space) whose elevation falls between β_0 and $\beta_0+\mathrm{d}\beta_0$ and are followed by a local maximum of elevation between β_1 and $\beta_1+\mathrm{d}\beta_1$ after a time lag between τ and $\tau+\mathrm{d}\tau$, *where both the local maximum at* $t=0$ *and the local maximum at* $t=\tau$ *must be two successive wave crests* (the subscript s.c. stands for *successive crests*). From the definition of $\mathrm{EX_c}$ and $\mathrm{EX_{s.c.}}$ it follows that

$$\mathrm{EX_{s.c.}}(\beta_0,\beta_1,\tau) \le \mathrm{EX_c}(\beta_0,\beta_1,\tau) \quad \frac{\mathrm{EX_{s.c.}}(\beta_0,\beta_1,\tau)}{\mathrm{EX_c}(\beta_0,\beta_1,\tau)} \to 0$$

$$\text{as} \quad \tau\to\infty.$$

As β_0 and $\beta_1\to\infty$, from Eq. (21) it has been proven that two successive wave crests lagged in time by $T_2^* + \delta\tau$ with $\delta\tau\in\delta\Gamma$ are, with probability approaching one, two local maxima lagged in time by $T_2^* + \delta\tau$. This implies

$$\mathrm{EX_{s.c.}}(\beta_0,\beta_1,\tau) = \begin{cases} \mathrm{EX_c}(\beta_0,\beta_1,\tau) & \tau = T_2^* + \delta\tau \quad \delta\tau\in\delta \\ 0 & \text{elsewhere} \end{cases}. \tag{23}$$

The exact expression for the joint probability density function $p(\beta_0, \beta_1)$ of two successive wave crests is given by

$$p(\beta_0,\beta_1) = \frac{\int_0^\infty \mathrm{EX_{s.c.}}(\beta_0,\beta_1,\tau)\mathrm{d}\tau}{\mathrm{EX}_+} \tag{24}$$

where EX_+ is defined as in Eq. (9). If β_0 and $\beta_1\to\infty$, since Eq. (23) holds, Eq. (24) simplifies as the following

$$p(\beta_0,\beta_1) \simeq \frac{1}{2\pi}\frac{\ddot{\eta}_c(0)\ddot{\eta}_c(T_2^*)}{1-\psi_{T_2^*}^2}\exp\left[-\frac{\beta_0^2+\beta_1^2-2\psi_{T_2^*}\beta_0\beta_1}{2(1-\psi_{T_2^*}^2)}\right] \int_{\delta\tau\in\delta\Gamma}\exp\left(-\frac{1}{2}K^*\delta\tau^2\right)\mathrm{d}(\delta\tau). \tag{25}$$

The integral that appears in Eq. (25) can bounded by $\int_{-\infty}^{\infty}\exp(-(1/2)K^*\delta\tau^2)d(\delta\tau) = \sqrt{2\pi}/\sqrt{K^*}$ obtaining the p.d.f.

$$p_\mathrm{a}(\beta_0,\beta_1) = \frac{1+\psi_2^*\ddot{\psi}_2^*}{\sqrt{-2\pi\ddot{\psi}_2^*(1-\psi_2^{*2})^3}} \exp\left[-\frac{\beta_0^2+\beta_1^2-2\psi_2^*\beta_0\beta_1}{2(1-\psi_2^{*2})}\right]\sqrt{(-\beta_0+s\beta_1)(-\beta_1+s\beta_0)} \tag{26}$$

where $\psi_2^* \equiv \psi_{T_2^*}$ and $\ddot{\psi}_2^* \equiv \ddot{\psi}_{T_2^*}$. From Eq. (26) the following upper bound for $p_\mathrm{a}(\beta_0, \beta_1)$ is readily derived

$$p_\mathrm{a}(\beta_0,\beta_1) \le \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\psi_2^*(1-\psi_2^{*2})}} \exp\left[-\frac{\beta_0^2+\beta_1^2-2\psi_2^*\beta_0\beta_1}{2(1-\psi_2^{*2})}\right]\sqrt{\beta_0\beta_1} \tag{27}$$

since $(\beta_0-s\beta_1)(\beta_1-s\beta_0)\le\beta_0\beta_1$ and for typical spectra one can prove that $(1+\psi_2^*\ddot{\psi}_2^*/\sqrt{-\ddot{\psi}_2^*(1-\psi_2^{*2})})\le 1$. Because the following asymptotic expansion for the modified Bessel function $I_0(y)$ holds

$$I_0(y) = \frac{1}{\sqrt{2\pi}}\frac{\exp(y)}{\sqrt{y}} + o(y^{-1}) \quad \text{as} \quad y\to\infty, \tag{28}$$

setting $y = k\beta_0\beta_1/(1-k^2)$ in Eq. (28), the upper bound (27) is the asymptotic expansion of the following bivariate Weibull distribution

$$p_\mathrm{W}(\beta_0,\beta_1) = \frac{\beta_0\beta_1}{1-k^2}\exp\left[-\frac{\beta_0^2+\beta_1^2}{2(1-k^2)}\right]I_0\left(\frac{k\beta_0\beta_1}{1-k^2}\right). \tag{29}$$

Here, the Weibull parameter is $k=\psi_2^*$. The bivariate Weibull distribution has been used by many authors to model the distribution of successive wave heights in narrow-band Gaussian seas [23–26] or the distribution of successive wave periods [27] and the parameter k is estimated as

$$k_{\mathrm{m}} = \sqrt{\frac{\left[\int_0^\infty S(\omega)\cos(\omega T_{\mathrm{m}})\mathrm{d}\omega\right]^2 + \left[\int_0^\infty S(\omega)\sin(\omega T_{\mathrm{m}})\mathrm{d}\omega\right]^2}{m_0}} \quad (30)$$

with $T_{\mathrm{m}} = 2\pi\sqrt{m_0/m_2}$ the mean zero up-crossing period. By means of the theory of quasi-determinism, a bivariate Weibull law has been derived as a model for the statistics of successive wave crests. The Weibull parameter k [see Eq. (29)] is equal to the non-dimensional parameter $\psi_2^* = \psi(T_2^*)/\psi(0)$ with T_2^* the abscissa of the second absolute maximum of the autocovariance function $\psi(T)$.

6. Validation

The probability laws $p_{\mathrm{a}}(\beta_0, \beta_1)$ and $p_{\mathrm{W}}(\beta_0, \beta_1)$, i.e. Eqs. (26) and (29), are now validated by performing Monte Carlo simulations with the following spectral form

$$S(\omega_{\mathrm{p}}) = \begin{cases} \dfrac{1}{\omega_{\max} - \omega_{\min}} & \omega_{\min} < \omega < \omega_{\max} \\ 0 & \text{elsewhere} \end{cases}. \quad (31)$$

Assuming $\omega_{\max} = 1.5\omega_{\mathrm{p}}$ and $\omega_{\min} = 0.5\omega_{\mathrm{p}}$, by means of Eq. (3), realizations of a Gaussian sea state with the given spectrum (31) have been generated, with roughly 90,000 waves. In Figs. 2–4, the theoretical probabilities of exceedance $\Pr[\beta_0 > x_0, \beta_1 > x_1]$ of the asymptotic p.d.f. (26) and the Weibull p.d.f. (29) are compared to the probabilities of exceedance derived from the Monte Carlo simulations. As one can see from the plots, the asymptotic $p_{\mathrm{a}}(\beta_0, \beta_1)$ and the Weibull $p_{\mathrm{W}}(\beta_0, \beta_1)$ are respectively a lower bound and an upper bound of the exact p.d.f. $p(\beta_0, \beta_1)$. The distribution $p_{\mathrm{a}}(\beta_0, \beta_1)$ converges to the exact distribution $p(\beta_0, \beta_1)$ for $\beta_0 > 2$ and $\beta_1 > 2$, whereas the convergence of $p_{\mathrm{W}}(\beta_0, \beta_1)$ is attained for $\beta_0 > 2.5$ and $\beta_1 > 2.5$.

7. Application: the maximum expected wave crest pressure in undisturbed deep water

Consider N_{c} consecutive waves of a sea state with a typical JONSWAP energy spectrum as in Eq. (5). The probability that the largest crest height of this set of N_{c} waves is smaller than a threshold h is equal to the probability that all N_{c} wave crests are smaller than h, i.e.

$$\Pr(C_{\max} \le h) = \Pr(C_1 \le h, C_2 \le h, \ldots, C_{N_{\mathrm{c}}} \le h). \quad (32)$$

Here, $C_{\max}$ is the largest wave crest of the set $\{C_1, C_2, \ldots, C_{N_{\mathrm{c}}}\}$. Assuming that the wave crest heights are stochastically independent of one another yields

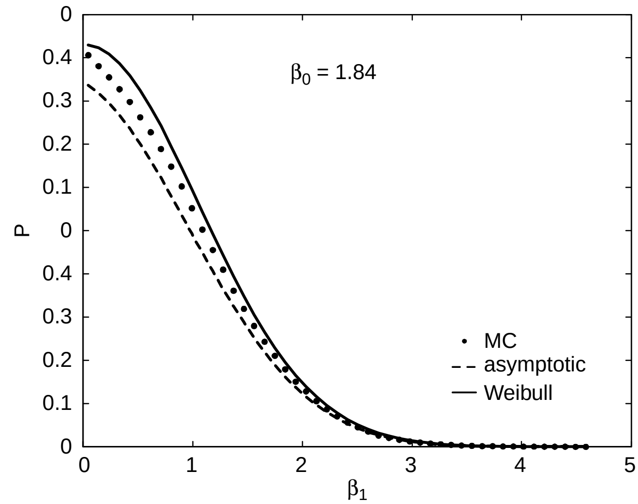

Fig. 2. The probabilities of exceedance: comparison among the asymptotic p.d.f. $p_{\mathrm{a}}(\beta_0, \beta_1)$, the Weibull $p_W(\beta_0, \beta_1)$ and the Monte Carlo simulations for $\beta_0 = 1.84$.

$$\Pr(C_{\max} \le h) = [\Pr(C_1 \le h)]^{N_{\mathrm{c}}}. \quad (33)$$

This expression underestimates the probability that all the wave crest heights are smaller than h, because it does not take into account the clustering effect, i.e. if a wave crest is smaller than h, neighboring crest heights will be more likely to be less as well due to their dependence. However, the clustering effect is not expected to involve many neighboring crests at the high level of h. In fact, for large h, the wave crest is the center of a wave group, where we can expect that one or two waves before, and one or two waves after, will also be higher than the mean wave crest (see [14], pp. 177–180). In the following, the probability that the next wave crest height C_{j+1} is less than h, is assumed to depend only upon whether the last wave crest height C_j was less, and not upon still earlier wave crest heights $C_{j-1}, C_{j-2}, \ldots$. This is a form of a Markov

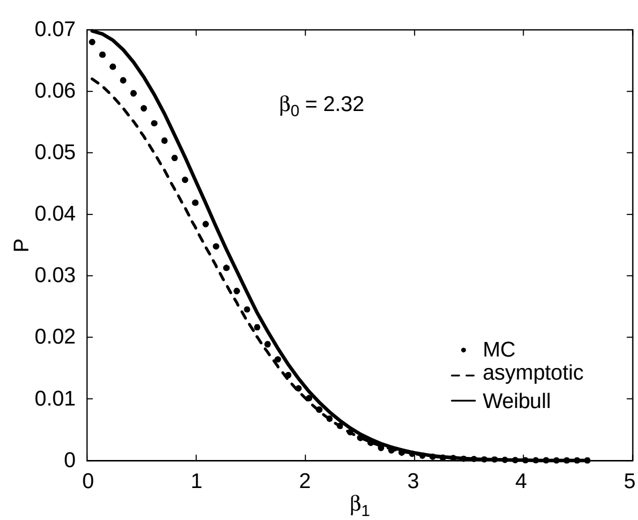

Fig. 3. The same probabilities of exceedance as in Fig. 2, for $\beta_0 = 2.32$.

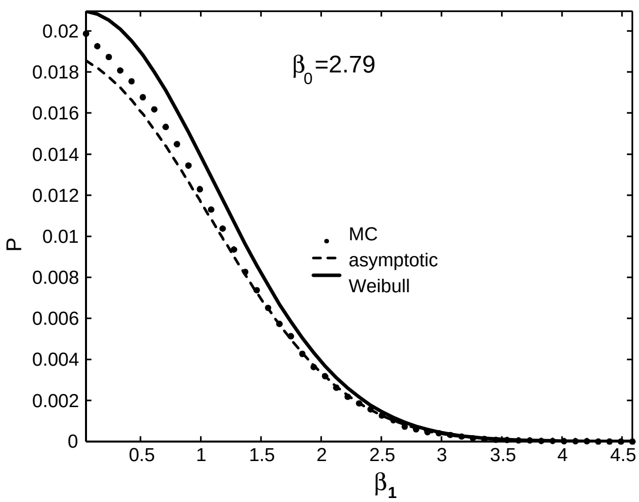

Fig. 4. The same probabilities of exceedance as in Fig. 1, for $\beta_0=2.79$.

chain (one-step memory in time), which gives for Eq. (32) the following simplification

$$\Pr(C_{\max}\le h)=\Pr(C_1\le h)\cdot\Pr(C_2\le h/C_1\le h)\ldots\Pr(C_{N_c}\le h/C_{N_c-1}\le h)=\Pr(C_1\le h)\quad[\Pr(C_j\le h/C_{j-1}\le h)]^{N_c-1}\quad(34)$$

where $\Pr(C_j\le h/C_{j-1}\le h)$ is the probability that a wave crest height is smaller than h given the condition that the precedent wave crest is less as well. The probability (34) can be easily computed since, from Eq. (29)

$$\Pr(C_j\le h/C_{j-1}\le h)=\frac{\int_0^{h/\sigma}\int_0^{h/\sigma}p_W(\beta_0,\beta_1)\mathrm{d}\beta_0\mathrm{d}\beta_1}{1-\exp\left[-\frac{1}{2}\left(\frac{h}{\sigma}\right)^2\right]}$$

and

$$\Pr(C_j\le h)=1-\exp\left[-\frac{1}{2}\left(\frac{h}{\sigma}\right)^2\right]\quad j=1,\ldots,N_c.$$

The maximum expected wave crest $\bar{C}_{\max}$ can then be evaluated as (see [14], pp. 177–180)

$$\bar{C}_{\max}=\int_0^\infty[1-\Pr(C_{\max}\le h)]\mathrm{d}h.$$

Observe that $\bar{C}_{\max}$ depends upon the choice of the parameter k of the Weibull distribution (29). As an application, consider the first-order random wave pressure in an undisturbed field on deep water at a fixed point in the sea, given by

$$\eta(z,t)=\frac{\Delta p(z,t)}{\rho g}=\sum_{i=1}^N a_i\exp\left(\frac{\omega_i^2}{g}z\right)\cos(\omega_i t+\varepsilon_i).\quad(35)$$

with $z\in[0,-\infty)$. For fixed z, by setting

$$\tilde{a}_i=a_i\exp\left(\frac{\omega_i^2}{g}z\right),$$

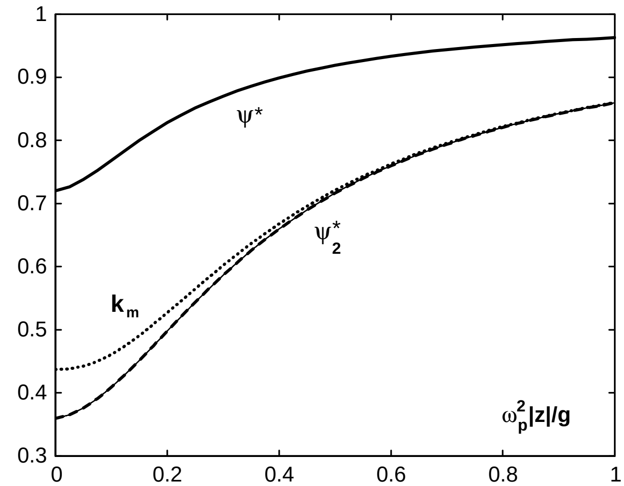

Fig. 5. Wave pressure in undistributed deep water: the parameters ψ^*, ψ_2^* and k_m as a function of the non-dimensional depth $\omega_p^2|z|/g$.

$\eta(z,t)$ is a stationary ergodic stochastic Gaussian process with spectrum

$$\tilde{S}(\omega;z)=S(\omega)\exp\left(\frac{\omega^2}{g}z\right).$$

Here, according to Eq. (4), $\tilde{S}(\omega;z)$ is defined as

$$\tilde{S}(\omega;z)d\omega=\sum_{\omega_i\in[\omega,\omega+d\omega]}\frac{\tilde{a}_i^2}{2}.$$

The autocovariance function $\psi(z,T)=\langle\eta(z,t)\eta(z,t+T)\rangle$ of $\eta(z,t)$ can be evaluated by the following integral

$$\psi(z,T)=\int_0^\infty\tilde{S}(\omega;z)\cos(\omega T)\mathrm{d}\omega$$

and the standard deviation of the wave pressure at level z is readily obtained as $\sigma=\sqrt{\psi(z,0)}$. In Fig. 5 the parameters ψ^*, ψ_2^* and k_m [see Eq. (30)] are plotted as a function of the dimensionless depth $\omega_p^2|z|/g$. In Fig. 6 the plots of

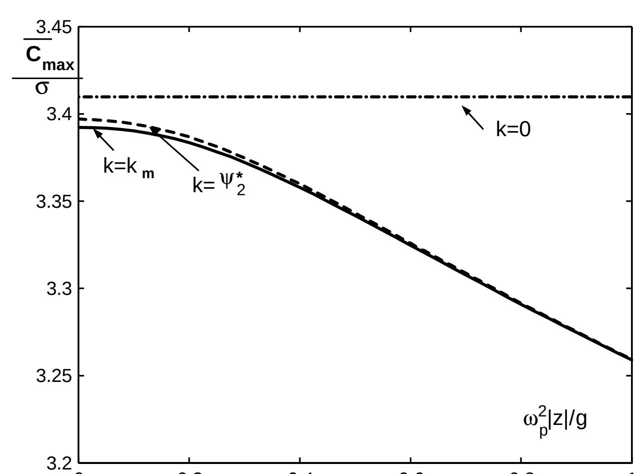

Fig. 6. The maximum expected wave crest pressure $k=\psi^*$, ψ_2^*, k_m as a function of the non-dimensional depth $\omega_p^2|z|/g$.

the maximum expected wave crest pressure $\bar{C}_{max}$, evaluated using $k=\psi_2^*$, $k=k_m$ and $k=0$ respectively ($N_c=200$ waves), are displayed. Note that the case of $k=0$ imposes stochastic independence among the wave crests, since the Weibull law (29) reduces down to the product of two Rayleigh laws. From Fig. 5 one can see that the wave pressure spectrum tends to become narrow as the depth level increases, since ψ^* tends to 1. This implies that the assumption of stochastic independence among the pressure crests breaks down. As a consequence, the maximum expected wave crest pressure $\bar{C}_{max}$ computed with $k=0$ overestimates the maximum expected wave crest pressure $\bar{C}_{max}$ evaluated using both $k=\psi_2^*$ and $k=k_m$. (see Fig. 6). Observe that the maximum expected wave crest pressure $\bar{C}_{max}$ computed with $k=\psi_2^*$ is slightly more conservative than the maximum expected wave crest pressure $\bar{C}_{max}$ computed with $k=k_m$, since $k_m>\psi_2^*$ (see Fig. 5).

8. Conclusions

The necessary and sufficient conditions for the occurrence of two very large successive wave crests are given. As a corollary, it is proven that the tail probability of the joint distribution of two successive wave crests is given by a bivariate Weibull law. Here, the Weibull parameter is equal to $\psi_2^*=\psi(T_2^*)/\psi(0)$ with T_2^* the abscissa of the second absolute maximum of the autocovariance function $\psi(T)$. It is also proven that the first two-peaks part of the autocovariance function $\psi(T)$ describes the structure of two successive-wave patterns. The theoretical results agree well with the Monte Carlo simulations. Finally, as an application, the maximum expected wave crest pressure in an undisturbed deep water waves is evaluated considering the stochastic dependence of successive wave crests.

Appendix

The joint probability in Eq. (19) is multivariate Gaussian and can be expressed as

$$p[\eta_0=\beta_0, \dot{\eta}_0=0, \dot{\eta}_\tau=0, \eta_\tau=\beta_1]$$
$$=\frac{1}{(2\pi)^2\sqrt{|D|}}\exp\left[-\frac{1}{2}f(\beta_0,\beta_1,\tau)\right].$$

Here, $f(\beta_0,\beta_1,\tau)=\boldsymbol{\omega} D^{-1}\boldsymbol{\omega}^t$ and D is the covariance matrix of the row vector of variables $\boldsymbol{\omega}=[\eta_0, \dot{\eta}_0, \dot{\eta}_\tau, \eta_\tau]$ defined as

$$D=\begin{bmatrix} 1 & 0 & \dot{\psi}_\tau & \psi_\tau \\ 0 & 1 & -\ddot{\psi}_\tau & -\dot{\psi}_\tau \\ \dot{\psi}_\tau & -\ddot{\psi}_\tau & 1 & 0 \\ \psi_\tau & -\dot{\psi}_\tau & 0 & 1 \end{bmatrix}.$$

By Taylor-expanding the function f with respect to the time lag τ, starting from $\tau=T_2^*$, yields

$$f(\beta_0,\beta_1,\tau)=\frac{\beta_0^2+\beta_1^2-2\psi_{T_2^*}\beta_0\beta_1}{1-\psi_{T_2^*}^2}+K^*\delta\tau^2+o(\delta\tau^2)$$

where

$$K^*=-\frac{\ddot{\psi}_{T_2^*}}{1-\ddot{\psi}_{T_2^*}^2}\ddot{\eta}_c(0)\ddot{\eta}_c(T_2^*).$$

The parameter $K^*\geq 0$, since the coefficient $(\ddot{\psi}_{T_2^*}/1-\ddot{\psi}_{T_2^*}^2)$ is always negative ($\ddot{\psi}_{T_2^*}<0$ because ψ attains a maximum at $t=T_2^*$ and $|\ddot{\psi}_{T_2^*}|\leq 1$ by definition) and $\ddot{\eta}_c(0)\ddot{\eta}_c(T_2^*)>0$ if the constraint (16) holds. Note that as $\beta_0\to\infty$ and $\beta_1\to\infty$ the coefficient K^* goes to infinity.

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