Weakly nonlinear statistics of high random waves

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It is well known that in a Gaussian sea state for an infinitely narrow spectrum the crest height and the trough depth follow the same Rayleigh distribution, because of linearity of the first order Stokes expansion solution. For spectra of finite bandwidth, Boccotti obtained, as a corollary of his first formulation of the theory of quasideterminism (which is exact to the first order in a Stokes expansion), that the crest height and the trough depth still follow asymptotically the Rayleigh law for high waves in Gaussian sea states. In this paper we extend the theory of quasideterminism of Boccotti to the second-order, deriving new wave crest and wave trough distributions that take into account nonlinear effects and are valid for finite bandwidth of the spectrum in deep water. Nonlinear Monte Carlo simulations validate our theoretical predictions and comparisons with experimental data and the recent model of Forristall are finally presented.
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I. INTRODUCTION

The first order Stokes solution of the free surface displacement is a random Gaussian process of time. Longuet-Higgins[1] showed that for an infinitely narrow spectrum the wave height follows the Rayleigh distribution. Because of the symmetry of a Gaussian sea state the crest and trough distributions follow the same Rayleigh law for narrow spectra. For the case of spectra of finite bandwidth, Boccotti[2–9] showed as corollary of his first formulation of the theory of quasideterminism that the crest height and the trough depth still follow asymptotically the same Rayleigh law for high wave amplitude (see also Lindgreen,[10,11] Maes and Breitung,[12] Breitung,[13] Sun,[14] Leadbetter and Rootzen,[15] and Kac and Slepian[16]).

If the nonlinear effects are not negligible, the probability density function $p(\eta)$ of the surface displacement tends to deviate from being Gaussian. In particular, second order nonlinearities make high crests to be more probable than deep troughs, i.e., the skewness of $p(\eta)$ is not zero (Longuet-Higgins[17]). Tayfun[18,19] and Tung and Huang[20] investigated the crest-trough symmetry, deriving the probability distributions of both the second order crest and trough under the hypothesis of narrow-band spectrum. Arena and Fedele[21] obtained the crest and the trough distributions of a general nonlinear narrow-band stochastic family, which includes many processes in the mechanics of the sea waves (either in an undisturbed field or in front of a vertical wall). Other second order models have been proposed by Al-Humoud *et al.*;[22] Wu and Song[23] derived the distribution of local maxima by solving for the joint distribution of the sea surface and its first derivative by the moment method. Moreover two models (Prevosto *et al.*,[24] Forristall[25]) were proposed for the crest height distribution of three dimensional waves: they give results very close to each other and in good agreement with field data (Prevosto and Forristall[26]).

Several authors have also investigated third order effects as the modulation instability (see Janssen[27] and references herein). It is believed that they can cause the occurrence of very large amplitude waves (freak waves). The mechanism that could generate a freak wave is related to the four-wave interaction (see Janssen,[27] Komen *et al.*[28]). Nonlinear energy transfer among nonresonant and resonant quartets is governed by the deterministic Zakharov integral differential equation.[29] Under the assumption of narrow-band spectrum, the Zakharov equation reduces down to the nonlinear Schrödinger equation (NLS). Trulsen *et al.*[30] proposed an enhanced NLS equation (DNLS) valid for broader spectral bandwidth and larger steepness. Using this equation, Trulsen and Dysthe[31] showed that a freak wave can be generated through nonlinear self-modulation of a slowly modulated wave train. Their numerical solution of the DNLS equation agrees very well with experimental record from the Draupner time series (see also Wist *et al.*[32] for details about this time series). Onorato *et al.*,[33] solving numerically both the NLS and DNLS equations, showed that the cumulative probability density function of the wave heights deviates from the Rayleigh distribution: high wave heights are more probable than when judged by the Rayleigh law. For the case of a JONSWAP spectrum, they found that the deviation from the Rayleigh law increases, as both the enhancement coefficient γ and the Phillips parameter increase (see also Onorato *et al.*[34]). Experimental work on the influence of the modulation instability on the cumulative probability function of wave crests has been presented by Stansberg.[35]

In this paper, unidirectional waves in deep water are considered. Based on the work of Boccotti[2–9] new analytical wave crest and trough distributions are derived. They take into account second-order effects and are valid for finite bandwidth of the spectrum in deep water (see also

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Longuet-Higgins[36] for an alternative analysis).

Boccotti showed that in a Gaussian sea state, if it is known that a very high local maximum (minimum)—very high with respect to the mean crest height (trough depth)—occurs in some time and location, this implies that a well-defined quasideterministic wave group generates the highest local maximum (minimum) which tends to be the crest (trough) of its wave. As corollary he derived that both the probabilities of exceeding the crest and trough amplitude follow asymptotically the Rayleigh distribution.

The main result of this paper is that second order analytical models for the prediction of extreme events can be derived by means of the theory of quasideterminism of Boccotti.[2–9] This theory can also be extended to consider third order nonlinearities as shown recently by Fedele,[37] but this will be discussed in a future paper.

The authors, starting from the general second order Stokes solution of the surface displacement for long-crested waves, showed that the amplitude of the nonlinear crest (trough) depends upon the linear crest (trough) amplitude. Thus the probability distributions of the nonlinear crest and trough are obtained.

Numerical distributions obtained by Monte Carlo simulations of nonlinear second order Gaussian sea states are in agreement with our analytical results. Comparisons with experimental data and with the recent model of Forristall[26,27] are finally discussed.

II. THE THEORY OF QUASIDETERMINISM

The theory of quasideterminism of Boccotti[2–9] is now presented for the case of unidirectional random waves in deep water (the direction is along the x axis). Similar analysis holds for the general case of short-crested random waves.

If in a Gaussian sea state it is known that a very high local maximum occurs in some location and time, this implies with high probability that a well-defined wave group generates the high local maximum. In detail, let us assume that a local wave maximum of given elevation h_0 occurs at a time $t=t_0$ at a fixed point $x=x_0$. If $h_0/\sigma\to\infty$, i.e., the crest is very high with respect to the mean crest height, then with probability approaching 1, the surface displacement at $x=x_0+X$ is asymptotically equal to the deterministic form

$$\overline{\eta_L}(X,T)=\frac{\Psi(X,T)}{\Psi(0,0)}h_0, \tag{1}$$

being σ the standard deviation of the free surface displacement $\eta(x,t)$. The space–time covariance $\Psi(X,T)$ is given by

$$\Psi(X,T)\equiv\langle\eta(x,t)\,\eta(x+X,t+T)\rangle, \tag{2}$$

where

$$\langle f(t)\rangle=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau}f(t)dt$$

is the time average. An exceptionally high local maximum, with a very high degree of probability, is also a wave crest of its wave, because the space–time covariance function $\Psi(X,T)$ attains its absolute maximum at $(X=0,T=0)$ as it will be shown below. A direct consequence is that the number of wave crests exceeding a fixed threshold b tends to coincide with the number of local wave maxima exceeding it, provided the fixed threshold is very high. As a consequence, the number of wave crests exceeding a very high threshold b tends to coincide with the number of b up-crossings (b_+), that is

$$\frac{N_{cr}(b;\Delta t)}{N_+(b;\Delta t)}\to 1\quad\text{as } b/\sigma\to\infty.$$

Here, $N_{cr}(b;\Delta t)$ and $N_+(b;\Delta t)$ denote, respectively, the number of wave crests exceeding the threshold b and the number of b_+ in the very large time interval Δt. Since (see Boccotti[9])

$$N_+(b;\Delta t)\propto\exp\left(-\frac{b^2}{2\sigma^2}\right)\Delta t$$

the probability of exceeding a wave crest height admits the following asymptotic expression:

$$P(h_0>b)=\frac{N_+(b;\Delta t)}{N_+(0;\Delta t)}=\exp\left(-\frac{b^2}{2\sigma^2}\right)\quad\text{as } b/\sigma\to\infty, \tag{3}$$

which is the well-known Rayleigh distribution.

The quasideterministic wave group $\overline{\eta_L}$: In physical space $\overline{\eta_L}(X,T)$ represents the evolution of a wave group which reaches its absolute maximum h_0 at time $t=t_0$ at the fixed point $x=x_0$. Set $S(\omega)$ as the unidirectional JONSWAP spectrum,

$$S(\omega)=Ag^2\omega^{-5}\exp\left[-\frac{5}{4}\left(\frac{\omega}{\omega_p}\right)^{-4}\right]\times\exp\left\{\ln\gamma\exp\left[-\frac{(\omega-\omega_p)^2}{2\chi^2\omega_p^2}\right]\right\}, \tag{4}$$

where ω_p is the peak frequency, A is the Phillips parameter and γ is the enhancement coefficient and $\chi=0.08$. As the parameter γ increases, the spectrum becomes higher and narrower around the spectral peak (see Onorato *et al.*[34]). For $\gamma=1$ and $A=0.0081$, Eq. (4) reduces down to the Pierson–Moskowitz spectrum. The jth order moment of the spectrum is

$$m_j=\int_0^{\infty}S(\omega)\omega^j\,d\omega.$$

In particular m_0 is the variance σ^2 of the surface displacement. According to the theory of sea states (see Boccotti[9]), to the first order in a Stokes expansion, the surface displacement $\eta(x,t)$ is a realization of the following stationary ergodic Gaussian process:

$$\eta(x,t)=\sum_{n=1}^{N}a_n\cos(k_nx-\omega_nt+\varepsilon_n). \tag{5}$$

Here, it is assumed that the frequencies ω_n are different from each other, the number N is infinitely large, and the phase angles ε_n, uniformly distributed in $[0,2\pi]$, are stochastically independent of each other. In deep water the wave numbers are given by $k_n=\omega_n^2/g$. Furthermore, all the amplitudes a_n are defined such that

$$\sum_j \frac{1}{2} a_j^2 = S(\omega)\Delta\omega \quad \text{with } \omega - \Delta\omega/2 < \omega_j < \omega + \Delta\omega/2. \tag{6}$$

This gives

$$\sigma^2 = \Psi(0,0) = \sum_{j=1}^{N} \frac{1}{2} a_j^2 = \sum_{j=1}^{N} S(\omega_j)\Delta\omega \overset{\Delta\omega\to 0}{=} \int_0^\infty S(\omega)d\omega$$

and the space–time covariance $\Psi(X,T)$ in Eq. (2) has the following expression (see Appendix B):

$$\Psi(X,T) = \int_0^\infty S(\omega)\cos(kX - \omega T)d\omega. \tag{7}$$

Let us observe that $\Psi(X,T)$ reaches its absolute maximum σ^2 at $(X=0, T=0)$. The surface displacement $\overline{\eta_L}$ is then given by

$$\overline{\eta_L}(X,T) = \pm \frac{h_0}{\sigma^2}\int_0^\infty S(\omega)\cos(kX - \omega T)d\omega, \tag{8}$$

where the plus and minus signs are for an initial condition at $t=t_0$ as a crest or a trough, respectively. Because of the symmetry of a Gaussian process, if a large trough occurs at a time $t=t_0$ at a fixed point $x=x_0$, with probability approaching 1 the surface displacement has a expression as Eq. (1) with a minus sign. In this case h_0 is the amplitude of the wave trough.

III. STATISTICS OF WEAKLY NONLINEAR WAVES

For the case of unidirectional waves in deep water analytical solutions for the probabilities of exceeding the second order nonlinear wave crest and trough are derived. A similar analysis also applies to the case of multidirectional waves, but this will not be discussed here.

A. Nonlinear free surface displacement in deep water for a given initial local maximum

The general second order solution for the surface displacements in deep water for long-crested waves (direction along the x axis) is[17,28]

$$\eta(x,t) = \sum_n a_n \cos\psi_n + \frac{1}{4}\sum_{n,m} a_n a_m[(k_n + k_m)\cos(\psi_n + \psi_m) - |k_n - k_m|\cos(\psi_n - \psi_m)]. \tag{9}$$

Here, $\psi_n = k_n(x_0 + X) - \omega_n T + \varepsilon_n$, $\{a_n\}_{n\in\aleph}$ are the amplitudes of the linear harmonics and $\{\varepsilon_n\}_{n\in\aleph}$ are undetermined phase angles and $x=x_0+X$. Assume that at $x=x_0$ a local maximum h occurs at time $t=t_0$. Then, the free surface displacement $\overline{\eta}(X,T)$ satisfies the following conditions:

$$\overline{\eta}|_{X=0,T=0} = h, \qquad \left.\frac{\partial\overline{\eta}}{\partial X}\right|_{X=0,T=0} = 0, \qquad \left.\frac{\partial^2\overline{\eta}}{\partial X^2}\right|_{X=0,T=0} < 0; \tag{10}$$

and admits the following expression (see Appendix A):

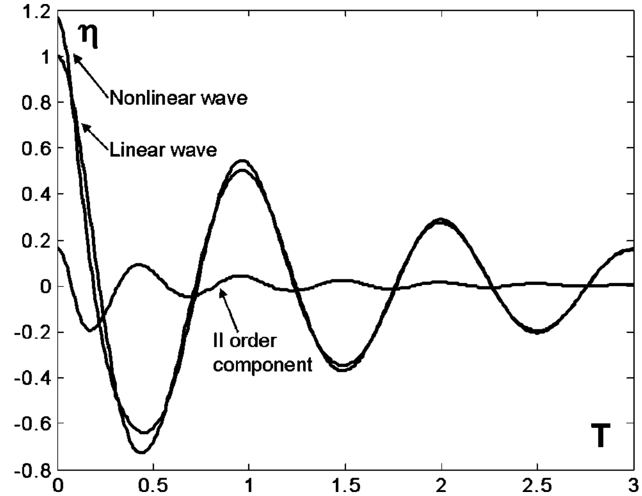

FIG. 1. The time evolution of the wave group $\overline{\eta}(X,T)$ and of its linear and nonlinear component at $X=0$.

$$\overline{\eta}(X,T) = \frac{h_0}{\sigma^2}\int_0^\infty S(\omega)\cos(g^{-1}\omega^2 X - \omega T)d\omega + \frac{h_0^2}{4g\sigma^4}\int_0^\infty\int_0^\infty S(\omega_1)S(\omega_2) \times\{(\omega_1^2+\omega_2^2)\cos[g^{-1}(\omega_1^2+\omega_2^2)X - (\omega_1+\omega_2)T] - |\omega_1^2-\omega_2^2|\cos[g^{-1}(\omega_1^2-\omega_2^2)X - (\omega_1-\omega_2)T]\}d\omega_1\, d\omega_2. \tag{11}$$

Here, according to Boccotti's theory, the amplitude h_0 of the linear wave crest is assumed to be very large if compared to the mean wave crest amplitude, i.e., $h_0/\sigma \to \infty$. Letting $k_p = \omega_p^2/g$, the wave number at the peak frequency, the ratio between the first term and the second term in Eq. (11) is of order $O(k_p h_0)$. Therefore, the expression of $\overline{\eta}(X,T)$ is valid as long as the nonlinear effects are weak, which means that the parameter $k_p h_0$ must be small. Defining the characteristic wave steepness as $\varepsilon_p = k_p\sigma$, one can write $k_p h_0 = \varepsilon_p(h_0/\sigma)$ and in the limit of $h_0/\sigma\to\infty$, the term $k_p h_0$ tends to zero if the steepness ε_p goes to zero as

$$\varepsilon_p \propto (h_0/\sigma)^{-1+\kappa}, \tag{12}$$

with κ a positive small number. This implies that, even if the linear crest h_0 is very large, one can always choose the steepness ε_p small enough so that the nonlinear effects are weak. In Fig. 1 a plot of the time evolution at $X=0$ of the wave group $\overline{\eta}(X,T)$ and of its linear and nonlinear component as well. As one can see, if the wave steepness is small, the nonlinear term in Eq. (11) does not modify the wave characteristics of the linear group, expect for a variation in the crest and trough amplitudes. Further studies are needed to investigate the validity of Eq. (11) in the context of a stochastic model for the space–time evolution of a wave in the neighborhood of a high crest, but this will not be discussed here. Instead, we shall show that from Eq. (11) one can derive new analytical expressions for the probabilities of exceeding both crest and trough amplitudes, which agree very well with both

Monte Carlo simulations of second order random seas and experimental data.

B. The nonlinear probabilities of exceeding the crest height and the trough depth

As $h_0/\sigma \to \infty$ the nonlinear crest amplitude h_C [from Eq. (11) for $X=0$ and $T=0$] is given by

$$h_C = h_0 + \frac{h_0^2}{4g\sigma^4}\int_0^\infty \int_0^\infty S(\omega_1)S(\omega_2)[(\omega_1^2+\omega_2^2) - |\omega_1^2-\omega_2^2|]d\omega_1\, d\omega_2 \tag{13}$$

and the nonlinear trough h_T has the following expression [see Eqs. (8) and (11)]:

$$h_T = h_0 - \frac{h_0^2}{4g\sigma^4}\int_0^\infty \int_0^\infty S(\omega_1)S(\omega_2)[(\omega_1^2+\omega_2^2) - |\omega_1^2-\omega_2^2|]d\omega_1\, d\omega_2. \tag{14}$$

Here, the nonlinear crest (trough) amplitude is a quadratic function of the linear crest (trough) amplitude h_0. Therefore the probabilities of exceeding $\Pr[h_C > h]$, $\Pr[h_T > h]$ are readily derived from the Rayleigh distribution of h_0 [see Eq. (3)]. We set the change of variables $w_1=\omega_1/\omega_p$, $w_2=\omega_2/\omega_p$, and $w=\omega/\omega_p$ and define the nondimensional spectrum

$$\tilde{S}(w) = \omega_p S(\omega_p w)/\sigma^2. \tag{15}$$

The variance σ_η of the second order surface displacement, is easily derived from Eq. (9) as

$$\sigma_\eta^2 = \frac{\sigma^2}{\beta^2}, \tag{16}$$

where

$$\frac{1}{\beta} = \sqrt{1+\frac{\varepsilon_p^2}{2}\int_0^\infty \int_0^\infty \tilde{S}(w_1)\tilde{S}(w_2)(w_1^4+w_2^4)dw_1\, dw_2}. \tag{17}$$

Assuming the dimensionless wave crest height as $\xi_{\text{crest}} = h_C/\sigma_\eta$, Eq. (13) can be written as follows:

$$\xi_{\text{crest}} = \beta u + \alpha(S)\beta u^2. \tag{18}$$

Here, the nondimensional coefficient $\alpha(S)$ is given by

$$\alpha(S) = \frac{\varepsilon_p}{4}\int_0^\infty \int_0^\infty \tilde{S}(w_1)\tilde{S}(w_2)[(w_1^2+w_2^2) - |w_1^2-w_2^2|]dw_1\, dw_2. \tag{19}$$

Because the random variable $u=h_0/\sigma$ has the Rayleigh distribution, the probability of exceeding the crest is readily derived, that is

$$P(\xi_{\text{crest}} > \xi) = \exp\left[-\frac{1}{8\alpha^2}\left(1-\sqrt{1+\frac{4\alpha\xi}{\beta}}\right)^2\right]. \tag{20}$$

Regarding the nonlinear wave trough depth h_T, setting the nondimensional wave trough as $\xi_{\text{trough}} = h_T/\sigma_\eta$, Eq. (14) yields

$$\xi_{\text{trough}} = \beta u - \alpha(S)\beta u^2, \tag{21}$$

where u has Rayleigh distribution as above. Some algebra gives the following expression for the probability of exceeding of the trough depth:

$$P(\xi_{\text{trough}} > \xi) = \left(\exp\left[-\frac{1}{8\alpha^2}\left(1-\sqrt{1-\frac{4\alpha\xi}{\beta}}\right)^2\right] - \exp\left[-\frac{1}{8\alpha^2}\left(1+\sqrt{1-\frac{4\alpha\xi}{\beta}}\right)^2\right]\right) \times [1-H(\xi-\beta/(4\alpha))], \tag{22}$$

where $H(x)$ is the step function.

The analytical distributions (20) and (22) are valid for $\xi \gg 1$ and wave steepness ε_p approaching zero as $\xi^{-1+\kappa}$ for $\xi \to \infty$ [see Eq. (12)], so that the nonlinear effects are weak as discussed in the preceeding section.

Note that these probabilities are consistent with the expression of the surface displacement $\bar{\eta}(X,T)$ in Eq. (11). For the case of a wave crest, $\bar{\eta}(X,T)$ gives the local space–time structure of a high crest that occurs at a specified location at a certain time (see Fig. 1 for the time evolution of the group at $X=0$). If the ratio h_0/σ is very large, i.e., $\xi \gg 1$, the linear component of the crest amplitude h_C [see Eq. (13)] tends to diverge, but the nonlinear component of h_C is always smaller than its linear counterpart since the wave steepness $\varepsilon_p \to 0$ as $\xi^{-1+\kappa}$ for $\xi \to \infty$. Moreover, from Eq. (20) large wave crests with $\xi \to \infty$ have a probability of occurrence approaching zero.

From Eqs. (18) and (21) it is evident that the dimensionless crest height $\xi_C = h_C/\sigma_\eta$ (trough depth $\xi_T = h_T/\sigma_\eta$) is greater (lower) than the linear part $\beta u = h_0/\sigma_\eta$ due to the term $\alpha(S)\beta u^2$. Therefore, as α increases, the crest ξ_C becomes steeper and the trough ξ_T becomes flatter than their respective linear counterpart h_0/σ_η. The parameter α is then a measure of the intensity of the nonlinear effects within the second order theory. To this order only the skewness of the surface elevation is influenced by the quadratic nonlinearities, but its kurtosis is almost equal to its Gaussian value. In fact, from Eq. (9) one can obtain

$$\frac{\langle \eta^3\rangle}{\sigma^3} = 6\alpha + O(\varepsilon_p^2), \quad \frac{\langle \eta^4\rangle}{\sigma^4} = 3 + O(\varepsilon_p^2). \tag{23}$$

The wave height $H = h_C - h_T = 2h_0$ is equal to the linear wave height, since the crest height increment is equal to the trough depth decrement, as one can see from Eqs. (13) and (14). Thus quadratic nonlinearities do not modify the linear wave height, which instead can increase due to cubic nonlinearities. In this case the kurtosis of the surface displacement can increase such that steeper crests and deeper troughs can occur with the same probability. Janssen[27] showed that, due to third order effects, the nonlinear energy transfer which occurs during a large crest event (freak wave) influences the probability density function of the surface displacement so that the kurtosis can reach values greater than its Gaussian value. As a consequence, the tail distribution gives increased probability of occurrence of large crest amplitudes, if compared to the case of the Rayleigh law. It is clear that both second and third order effects modify the tail distribution of the surface elevation, but the physical mechanisms which cause such deviations from the Gaussian law are different. Second order effects are due to the so-called bound harmonics, i.e., harmonics that do not satisfy the linear dispersion relation. They modify the wave crest and trough so that the skewness of the surface displacement is nonzero, but the kurtosis is almost Gaussian. Third order effects, instead are due to the four-wave interaction among free harmonics, i.e., harmonics which satisfy the linear dispersion relation. In this case the nonlinear energy transfer modifies the wave height so that both the wave crests and troughs increase in amplitude. No crest–trough asymmetry occurs as in the case of quadratic interaction.

Recently Guedes Soares *et al.*[38,39] have shown that in the sea state where abnormal waves occur the kurtosis is very high while the skewness is not so high. This indicates that third order nonlinear effects need to be considered since from Eq. (23) second order time series have skewness of order $O(\varepsilon_p)$ and kurtosis almost Gaussian. More realistic probabilistic model for freak waves should take into account both quadratic and cubic interactions in the wave evolution, as in the model proposed by Trulsen and Dysthe.[30,31] We point out that although the analytical probabilities in Eqs. (20)–(22) take into account only second order effects, the theory of quasideterminism of Boccotti[2–9] can be extended to consider third order nonlinearities,[37] but this will be discussed elsewhere.

C. The JONSWAP spectrum

In this section, the attention is focused on understanding how the spectral parameters γ and the Phillips parameter A of the JONSWAP spectrum influence the probability of occurrence of a wave crest according to the second order model (20). Onorato *et al.*[34] solved the time-like NLS (TNLS) equation by numerical techniques and observed that, as the coefficient γ and the Phillips parameter A increases, the nonlinearities becomes more important and the probability of the formation of the freak waves increases. Moreover, in Onorato *et al.*,[33] it is shown that the probability of exceeding the wave height tends to deviate from being Gaussian according to the TNLS equation or to DNLS equation proposed by Trulsen and Dysthe.[30,31] In these simulations second order effects have been neglected, therefore the crest and trough distributions are nearly the same.

For the case of quadratic nonlinearities, the wave height is not modified, but the crest amplitude increases and the trough depth decreases. For the JONSWAP spectrum [see Eq. (4)] the dimensionless spectrum $\tilde{S}(w)$ [see Eq. (15)] is given by

$$\tilde{S}(w) = \frac{F(w)}{\int_0^\infty F(w)dw}, \tag{24}$$

where

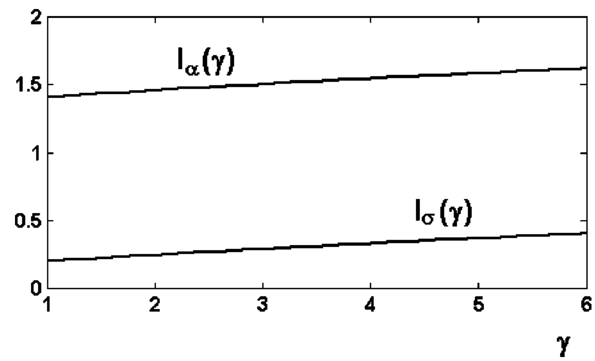

FIG. 2. The functions $I_\alpha(\gamma)$ and $\varepsilon_p^2/A = I_\sigma(\gamma)$.

$$F(w) = w^{-5}\exp\left[-\frac{5}{4}w^{-4}\right]\exp\left\{\ln\gamma\exp\left[-\frac{(w-1)^2}{2\chi^2}\right]\right\}. \tag{25}$$

The variance σ^2 can be computed by the expression

$$\sigma^2 = Ak_p^{-2}I_\sigma(\gamma), \tag{26}$$

where the function $I_\sigma(\gamma)$ is defined as

$$I_\sigma(\gamma) = \int_0^\infty F(w)dw. \tag{27}$$

Note that the Phillips coefficient A is related to the wave steepness ε_p as $\varepsilon_p^2 = AI_\sigma(\gamma)$. If one considers A and γ as the free parameters of the spectrum, the coefficient α can be written as

$$\alpha = \sqrt{A}I_\alpha(\gamma). \tag{28}$$

Here,

$$I_\alpha(\gamma) = I_1(\gamma)\sqrt{I_\sigma(\gamma)}, \tag{29}$$

where we have defined the integral

$$I_1(\gamma) = \frac{\int_0^\infty\int_0^\infty F(w_1)F(w_2)[(w_1^2+w_2^2) - |w_1^2 - w_2^2|]dw_1dw_2}{4\left(\int_0^\infty F(w)dw\right)^2}. \tag{30}$$

Both the functions $I_\alpha(\gamma)$ and $\varepsilon_p^2/A = I_\sigma(\gamma)$ increase monotonically with γ as it is shown in Fig. 2. This implies that [see Eq. (28)] the parameter α increases as both A and γ increase. As a consequence, for the case of quadratic nonlinearities, the probability of occurrence of large crest (trough) amplitudes increases (decreases) as both the enhancement coefficient γ and the Phillips coefficient A increase, but the wave height still follows the Rayleigh distribution.

Note that if one chooses the wave steepness ε_p and γ as the free parameters of the spectrum, the parameter α can be written as follows:

$$\alpha = \varepsilon_p I_1(\gamma). \tag{31}$$

Here (see Fig. 3), the function $I_1(\gamma)$ decreases monotonically with γ for fixed steepness ε_p. Therefore the coefficient α

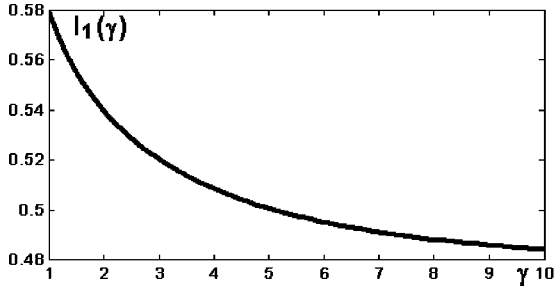

FIG. 3. The function $I_1(\gamma)$.

decreases as γ increases for fixed values of ε_p and it is linearly proportional to the steepness ε_p. Let us note from Fig. 3 that $I_1(\gamma)$ weakly varies as γ changes, implying that

$$0.48\varepsilon_p < \alpha < 0.58\varepsilon_p. \tag{32}$$

IV. VALIDATION

In this section we shall validate the distribution laws (20) and (22) by specializing to the case of rectangular spectra. The following spectral form is considered:

$$\tilde{S}(w) = \begin{cases} \dfrac{1}{w_{\max} - w_{\min}}, & w_{\min} < w < w_{\max}, \\ 0 & \text{elsewhere}, \end{cases} \tag{33}$$

with $1 \geqslant w_{\min} \geqslant 0, w_{\max} \geqslant 1$, where the dominant frequency is at $w=1$ and the mean period $T_m = 2\pi/\omega_m$ $[\omega_m = \omega_p(w_{\min} + w_{\max})/2]$ coincides with the dominant period $T_p = 2\pi/\omega_p$. The variance σ^2 can be chosen by assigning the wave steepness ε_p. The parameter α can be evaluated explicitly by solving analytically the double integral in Eq. (19), obtaining the following expression:

$$\alpha(w_{\min}, w_{\max}) = \varepsilon_p \frac{(w_{\min} + w_{\max})^2 + 2w_{\min}^2}{12}. \tag{34}$$

Note that for narrow-band spectrum

$$\alpha_\infty = \lim_{w_{\min}\to 1, w_{\max}\to 1} \alpha = \varepsilon_p/2 \tag{35}$$

in agreement with the narrow-band probability of exceeding (see, for example, Arena and Fedele[21]). In order to validate the new expressions for the probabilities of exceeding the nonlinear crest and the nonlinear trough, we have performed Monte Carlo simulations: we have used Eq. (9) to generate realizations of a non-Gaussian sea state with the given spectrum (33), with roughly 50 000 waves. It is assumed $w_{\min} = 0.50$, $w_{\max} = 1.50$. We have chosen $\varepsilon_p = 0.10$ for the simulations, which yields for α, β the following values:

$$\alpha = 0.038, \quad \beta = 0.992. \tag{36}$$

In Fig. 4 the plots of the theoretical curves [see also Eqs. (20) and (22)] are compared against the probabilities of exceeding derived from the Monte Carlo simulations and the relative narrow-band distributions as well ($\alpha_\infty = 0.050$, $\beta_\infty = 0.995$). The probabilities derived from the simulations agree well with the analytical probabilities. Observe that the nonlinear effects are less intense than the narrow-band case: this is due to the particular choice of the spectrum (33) for which $\alpha < \alpha_\infty$.

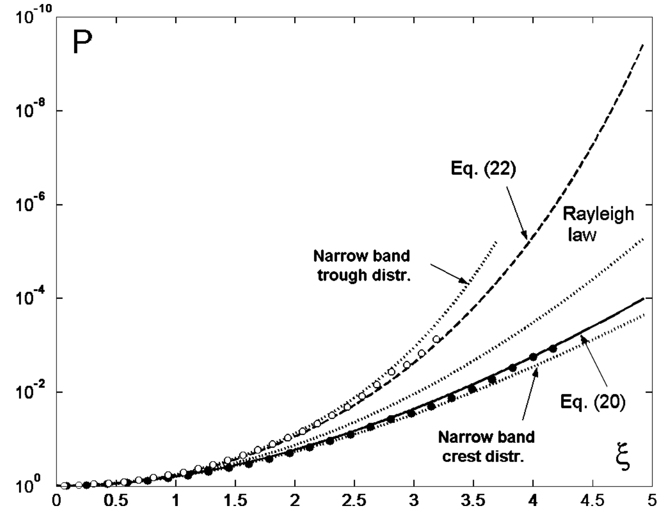

FIG. 4. Comparison between Rayleigh law, narrow-band second-order distributions, finite-band second-order distributions [Eqs. (20) and (22)] and data by numerical simulations with rectangular spectrum.

Finally, we have performed second-order simulations with a mean JONSWAP spectrum (with parameters $\gamma = 3.3$ and $A = 0.012$). The results are shown in Fig. 5. As one can see the agreement with the analytical model is quite good.

V. COMPARISONS WITH EXPERIMENTAL DATA

Comparisons will now be made with the data of the wave elevation measured at the Draupner field in the central North Sea, during the storms in the period from December 31, 1994 to January 20, 1995. Wist *et al.*[32] provided joint frequency tables of successive wave crest heights and wave trough depths of the Draupner time series, so that the empirical distributions are readily obtained. The peak frequency is $\omega_p = 0.55$ rad/s and mean wave period T_m has a value of

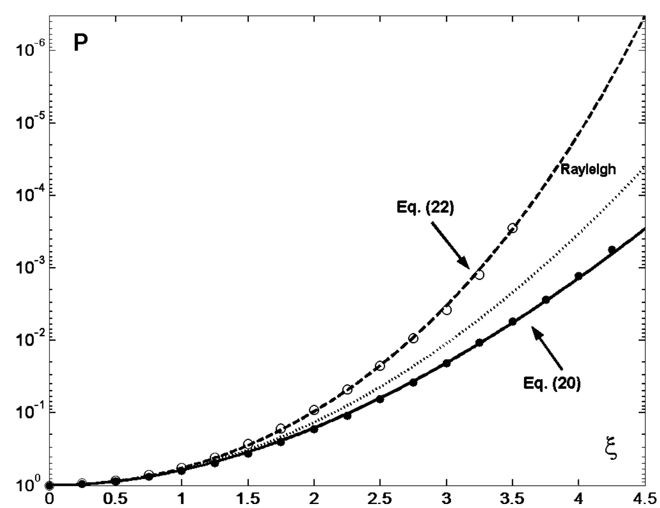

FIG. 5. Comparison between Rayleigh law, narrow-band second-order distributions, finite-band second-order distributions [Eqs. (20) and (22)] and data by numerical simulations with mean JONSWAP spectrum.

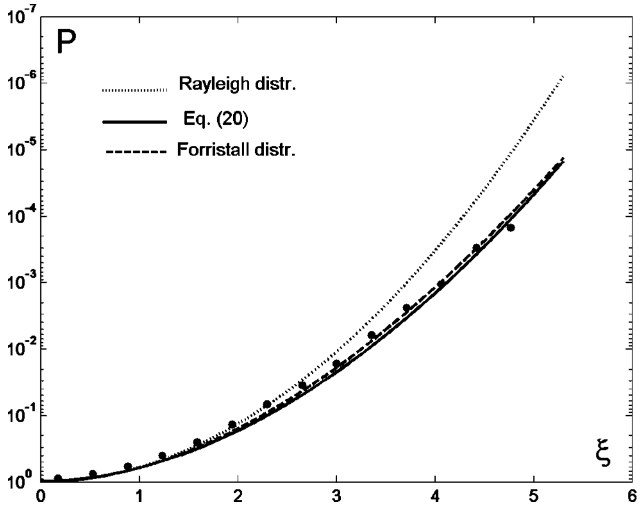

FIG. 6. The probability of exceeding the crest height: comparison between Forristall model and the proposed analytical model [Eq. (20)]. Experimental data from Draupner time series.

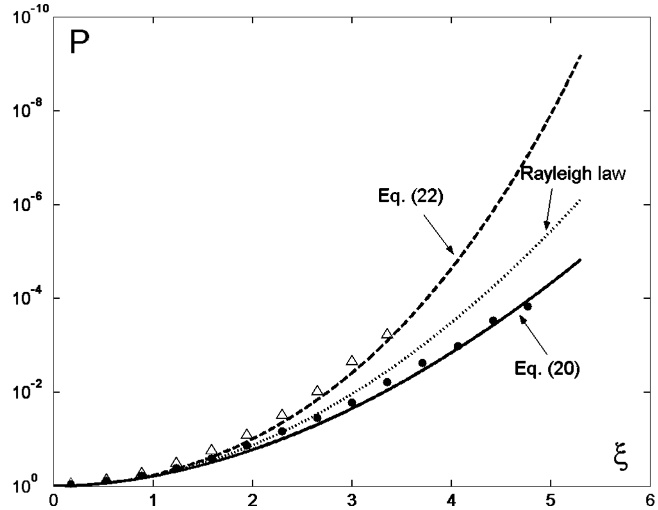

FIG. 7. Comparisons with experimental data from Draupner time series.

9.1 s. The significant wave height h_s is between 6.0 and 8.0 m and the wave steepness ε_p is in the range 0.05–0.06. The average spectral density of all the wave data corresponds to a JONSWAP spectrum with peakedness $\gamma=1.8$.

Wist *et al.* compared both the 2D and 3D models of Forristall[24–26] against the Draupner data. They did not find significant differences between the two models (see Fig. 6 in Ref. 32). Applying Forristall model for 2D long-crested random waves, his analytical distribution for the Draupner time series is the following:

$$P_F(\xi_{\text{crest}} > \xi) = \exp\left[-\left(\frac{\xi}{4\beta_1}\right)^{\beta_2}\right], \tag{37}$$

where $\beta_1=0.370$, $\beta_2=1.886$ see Table 3 in Ref. 32). By considering a JONSWAP spectrum with the above characteristics with $\varepsilon_p=0.06$, the parameters of the new crest distribution (16) are $\alpha=0.034$, $\beta=0.988$ for $\gamma=1.8$. In Fig. 6 the plots of these two distributions are compared against the Draupner data. The proposed distribution (20) agrees well with Forristall model as expected, because the latter is based on second order simulations.[25] Since the steepness is small, choosing a different value of $\gamma\in[1,10]$ gives $0.029<\alpha<0.035$ [see Eq. (32)]. In this range of values of α, the corresponding distribution curves are almost indistinguishable from the curve for $\gamma=1.8$ ($\alpha=0.034$). Therefore, within the second order theory, for unidirectional waves in deep water, the effects due to a finite-band spectrum are negligible. The empirical and analytical distributions [see Eqs. (20) and (22)] are plotted in Fig. 7. As one can see, the analytical curves compare well with the experimental data, but they are not able to fully capture the empirical tail distribution which could be due to both the freak wave event and statistical confidence.

VI. CONCLUSIONS

New analytical expressions for the probabilities of exceeding crest height and trough depth in a non-Gaussian sea state have been derived based on the theory of quasideterminism of Boccotti. The proposed distributions consider second order nonlinearities due to finite-band spectra in deep water. Monte Carlo simulations of nonlinear sea states with both rectangular and JONSWAP spectra have been performed to validate the proposed analytical probabilities. The agreement with the recent second order model of Forristall is quite good and the comparison with the Draupner data set has shown that the proposed model does not fully capture the empirical tail distribution which could be due to the freak wave event or statistical confidence. The main result of this paper is that analytical models for the prediction of nonlinear extreme events can be derived by means of the theory of quasideterminism of Boccotti.

APPENDIX A

Consider the assigned height h expanded as

$$h = h_0 + h_1 + h_2 + \cdots, \tag{A1}$$

where $h_0, h_1, h_2, \ldots$ are unknown parameters to be determined. We assume that $h_0 \propto \sigma$, $h_1 \propto \sigma^2, \ldots,$ $h_n \propto \sigma^{n+1}, \ldots,$ where σ is the standard deviation of the surface displacement. From the general solution (9) the two conditions in Eq. (10) give the following equations:

$$h_0+h_1+h_2+\cdots$$
$$=\sum_{n=1}^{N} a_n\cos\vartheta_n+\frac{1}{4}\sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m[(k_n+k_m)\cos(\vartheta_n+\vartheta_m)-|k_n-k_m|\cos(\vartheta_n-\vartheta_m)], \tag{A2}$$

$$0=-\sum_{n=1}^{N} a_n k_n\sin\vartheta_n+\frac{1}{4}\sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m[-(k_n+k_m)^2\sin(\vartheta_n+\vartheta_m)+|k_n-k_m|(k_n-k_m)\sin(\vartheta_n-\vartheta_m)], \tag{A3}$$

where $\vartheta_n=k_n x_0-\omega_n t_0+\varepsilon_n$. If one assumes $a_n\propto\sigma$, only the first two terms in the h expansion are nonzero. All the terms higher than the second order vanish. To the first order, Eqs. (A2) and (A3) give, respectively,

$$O(\sigma),\quad h_0=\sum_{n=1}^{N} a_n\cos\vartheta_n,\quad 0=\sum_{n=1}^{N} a_n k_n\sin\vartheta_n. \tag{A4}$$

The second equation in (A4) is satisfied if $\vartheta_n=0\ \forall n$ for any values of the coefficients $\{a_n\}_{n\in\aleph}$. This solution is not unique, since solutions with nonzero phases ϑ_n exist for particular choices of the coefficients $\{a_n\}_{n\in\aleph}$. Owing to the quasideterminism theory by Boccotti,[2–4,6,9] we shall prove that the condition $\vartheta_n=0\ \forall n$ is necessary and sufficient in order to have a wave crest. In fact, if $\vartheta_n=0\ \forall n$ the first equation in (A4) gives

$$h_0=\sum_n a_n, \tag{A5}$$

which is the highest value that h_0 can reach for an assigned discrete spectrum $\{a_n\}_{n\in\aleph}$. Therefore the condition $\vartheta_n=0\ \forall n$ implies that an absolute maximum is reached at a fixed point $x=x_0$ at time instant $t=t_0$ by the first order solution.

From Boccotti's theory if a very large crest height h_0 occurs at a fixed point $x=x_0$ at time instant $t=t_0$, with probability approaching 1, the free surface displacement [see Eq. (8)] in discrete form is given by

$$\overline{\eta_L}(X,T)=\sum_{n=1}^{N}\tilde{a}_n\cos\tilde{\psi}_n, \tag{A6}$$

where

$$\tilde{a}_n=\frac{h_0}{\sigma^2}S(\omega_n)\Delta\omega \tag{A7}$$

and

$$\tilde{\psi}_n=k_n X-\omega_n T. \tag{A8}$$

Because the wave group $\overline{\eta_L}(X,T)$ attains a maximum at $(X=0,T=0)$, it follows that

$$\overline{\eta_L}(X=0,T=0)=h_0\Rightarrow\sum_{n=1}^{N}\tilde{a}_n=h_0 \tag{A9}$$

and

$$\left.\frac{\partial\overline{\eta_L}}{\partial X}\right|_{X=0,T=0}=0\Rightarrow\sum_{n=1}^{N}\tilde{a}_n\sin\tilde{\psi}_n=0 \tag{A10}$$

(note that $\tilde{\psi}_n=0\ \forall n$ at $X=0$, $T=0$). Moreover, the second order derivative

$$\left.\frac{\partial^2\overline{\eta_L}}{\partial X^2}\right|_{X=0,T=0}=-\sum_n a_n k_n^2\cos\tilde{\psi}_n=-\sum_n a_n k_n^2<0$$

is always less than zero, confirming the existence of a local maximum, which is also the absolute maximum. Equations (A9) and (A10) are identical to Eqs. (A4) if $\tilde{\psi}_n=\psi_n$, $\tilde{a}_n=a_n$, which implies

$$\vartheta_n=0\ \ \forall n\quad\text{and } a_n=\frac{h_0}{\sigma^2}S(\omega_n)\Delta\omega. \tag{A11}$$

Thus the condition $\vartheta_n=0\ \forall n$ is sufficient and necessary in probability to guarantee that the linear component of the nonlinear wave group $\overline{\eta}$ attains a very large maximum at $X=0$, $T=0$.

To the second order, Eqs. (A2) and (A3) give

$$O(\sigma^2)\begin{cases} h_1=\dfrac{1}{4}\displaystyle\sum_{n,m} a_n a_m[(k_n+k_m)\cos(\vartheta_n+\vartheta_m)-|k_n-k_m|\cos(\vartheta_n-\vartheta_m)],\\ 0=\dfrac{1}{4}\displaystyle\sum_{n,m} a_n a_m[-(k_n+k_m)^2\sin(\vartheta_n+\vartheta_m)+|k_n-k_m|(k_n-k_m)\sin(\vartheta_n-\vartheta_m)].\end{cases} \tag{A12}$$

Since it is $\vartheta_n=0\ \forall n$, the second equation in Eq. (A12) is an identity, while the first equation gives the second order term of the wave crest

$$h_1=\frac{1}{4}\sum_{n,m} a_n a_m[(k_n+k_m)-|k_n-k_m|]. \tag{A13}$$

Note that, at $X=0$, $T=0$ the second order spatial derivative,

$$\left.\frac{\partial^2\overline{\eta}}{\partial X^2}\right|_{X=0,T=0}=-\sum_n a_n k_n^2\cos\vartheta_n-\frac{1}{4}\sum_{n,m} a_n a_m[(k_n+k_m)^3\cos(\vartheta_n+\vartheta_m)-|k_n-k_m|^3\cos(\vartheta_n-\vartheta_m)]$$

if $\vartheta_n=0\ \forall n$ reduces down to

$$\left.\frac{\partial^2\overline{\eta}}{\partial X^2}\right|_{X=0,Y=0,T=0} = -\sum_n a_n k_n^2 - \frac{1}{4}\sum_{n,m} a_n a_m[(k_n+k_m)^3 - |k_n-k_m|^3].$$

Here, this expression is always less than zero since $(k_n+k_m)^3-|k_n-k_m|^3>0$, confirming the existence of a local maximum which is also the absolute maximum. By considering Eq. (A11), that is $a_n=h_0/\sigma^2 S(\omega_n)\Delta\omega$, we obtain, in continuous form

$$h_1 = \frac{h_0^2}{4\sigma^4}\int_0^\infty\int_0^\infty S(\omega_1)S(\omega_2)[(k_1(\omega_1)+k_2(\omega_2)) - |k_1(\omega_1) - k_2(\omega_2)|]d\omega_1 d\omega_2. \quad \text{(A14)}$$

Here, the wave numbers k_1,k_2 are given by

$$k_1(\omega_1) = \omega_1^2/g, \quad k_2(\omega_2) = \omega_2^2/g.$$

Finally, we have that, if a very large crest height occurs, the second order height may be written as

$$h = h_0 + \frac{h_0^2}{4\sigma^4}\int_0^\infty\int_0^\infty S(\omega_1)S(\omega_2)[(k_1(\omega_1)+k_2(\omega_2)) - |k_1(\omega_1) - k_2(\omega_2)|]d\omega_1\, d\omega_2 + o(\sigma^2). \quad \text{(A15)}$$

More in general, the second order free surface displacement, when a very high crest occurs at time instant t_0 at point x_0 is given by

$$\overline{\eta}(X,T) = \frac{h_0}{\sigma^2}\int_0^\infty S(\omega)\cos(k(\omega)X-\omega T)d\omega + \frac{h_0^2}{4g\sigma^4}\int_0^\infty\int_0^\infty S(\omega_1)S(\omega_2)\{(k_1(\omega_1) + k_2(\omega_2))\cos[(k_1(\omega_1)+k_2(\omega_2))X - ((\omega_1 + \omega_2)T)] - |(k_1(\omega_1)-k_2(\omega_2))|\cos[(k_1(\omega_1) - k_2(\omega_2))X - (\omega_1-\omega_2)T]\}d\omega_1\, d\omega_2, \quad \text{(A16)}$$

where $k(\omega)=\omega^2/g$.

APPENDIX B

Using Eq. (5), the spatial–time covariance $\Psi(X,T)$ can be written as

$$\Psi(X,T) \equiv \langle\eta(x,t)\eta(x+X,t+T)\rangle = \sum_{i=1}^N\sum_{j=1}^N A_{ij}a_i a_j\cos(k_jX-\omega_jT) - \sum_{i=1}^N\sum_{j=1}^N B_{ij}a_i a_j\sin(k_jX-\omega_jT), \quad \text{(B1)}$$

where

$$A_{ij} = \langle\cos[\omega_i t-\tilde{\varepsilon}_i]\cos[\omega_j t-\tilde{\varepsilon}_j]\rangle,$$
$$B_{ij} = \langle\cos[\omega_i t-\tilde{\varepsilon}_i]\sin[\omega_j t-\tilde{\varepsilon}_j]\rangle, \quad \text{(B2)}$$

and $\tilde{\varepsilon}_j=k_jx_0$. Since $A_{ij}=\delta_{ij}/2$ and $B_{ij}=0$, in Eq. (B1) the following simplification holds:

$$\Psi(X,T) \equiv \sum_{j=1}^N \frac{1}{2}a_j^2\cos(k_jX-\omega_jT).$$

According to the definition of wave spectrum in Eq. (6), the continuous form of $\Psi(X,T)$ is

$$\Psi(X,T) = \int_0^\infty S(\omega)\cos(kX-\omega T)d\omega. \quad \text{(B3)}$$

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