A COMPLETE SET OF EIGENFUNCTIONS FOR THE STABILITY OF PULSATILE PIPE FLOW

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with

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LINEARIZED NON-NORMAL OPERATORS

$$\frac{\partial u}{\partial t} = \mathbf{L}u + \cancel{\mathcal{N}(u)}$$

Initial conditions

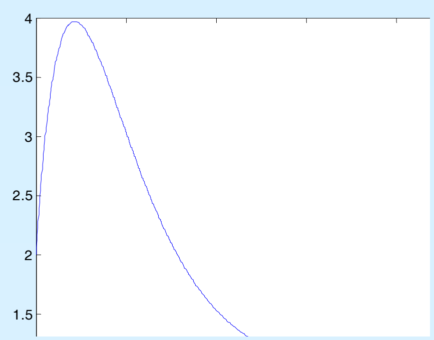
$$u_n \sim a_n \phi_n(r) \exp(-\lambda_n t) \quad \to 0 \quad as \quad t \to \infty$$

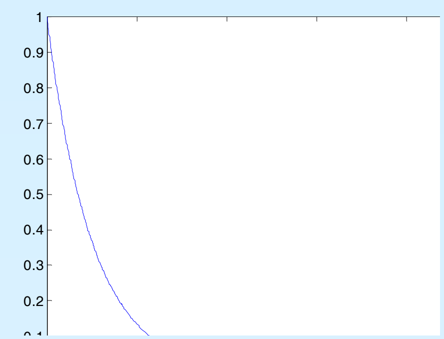
NON-NORMALITY (LL*≠L*L)

Set of non orthogonal eigenmodes $\langle \phi_n, \phi_m \rangle \neq \delta_{nm}$

$$E(t) \sim \sum_{n,m} \langle u_n, u_m^* \rangle = \underline{\sum_{n,m} a_n a_m \langle \phi_n, \phi_m \rangle e^{-(\lambda_n + \lambda_m^*)t}} \neq \underline{\sum_n a_n^2 e^{-(\lambda_n + \lambda_n^*)t}}$$

" the bump"





LINEAR STABILITY OF PULSATILE FLOW

Basic flow

$$\frac{\partial W}{\partial t} - \upsilon \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) = \frac{1}{\rho}\left(P_0 + K_\omega \cos \omega t\right)$$

$$W(r = 0, t) < \infty$$

$$W(r = R, t) = 0$$

Perturbation

$$u_r = u(z, r, t)$$

$$u_\theta = 0$$

$$u_z = W(r, t) + w(z, r, t)$$

$$\tilde{p} = p(z, r, t) + P(z, t)$$

Navier-Stokes

Equations

$$\frac{Du_r}{Dt} = \frac{u_\theta^2}{r} - \frac{1}{\rho}\frac{\partial \tilde{p}}{\partial r} + \upsilon\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{Du_\theta}{Dt} = -\frac{1}{\rho r}\frac{\partial \tilde{p}}{\partial \theta} + \upsilon\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z} + \upsilon \nabla^2 u_z$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

THE ORR-SOMMERFELD EQUATION

ELIMINATE PRESSURE

$$\left\{\begin{array}{l}
\frac{\partial u}{\partial t} + W\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \upsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right) \\
\frac{\partial w}{\partial t} + u\frac{\partial W}{\partial r} + W\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) - \frac{\partial^2 w}{\partial z^2}\right) \\
\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0
\end{array}\right.$$

$$\Psi(r,z,t) = \psi(r,t)e^{i\alpha z}$$

Stream function

$$u = -\frac{1}{r}\frac{\partial \Psi}{\partial z} = -\frac{\psi(r,t)}{r}i\alpha e^{i\alpha z} \qquad w = \frac{1}{r}\frac{\partial \Psi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial r}e^{i\alpha z}$$

$$\mathcal{L}\psi_t - Wi\alpha^3\psi + i\alpha(-\psi\mathcal{L}W + W\mathcal{L}\psi) = \upsilon\mathcal{L}^2\psi$$

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \alpha^2$$

INTRINSIC TIME SCALES

$$\tau_{\text{inertial}} = \frac{1}{\omega} \qquad \tau_{\text{viscous}} = \frac{\rho R^2}{\mu}, \qquad \tau_{\text{convection}} = \frac{L}{U_0}$$

$$\frac{\tau_{\text{inertial}}}{\tau_{\text{viscous}}} = \frac{\mu}{\rho\omega R^2} = \frac{1}{\gamma\psi^2}.$$

$$\frac{\tau_{\text{convection}}}{\tau_{\text{inertial}}} \sim L \to \infty \qquad \frac{\tau_{\text{convection}}}{\tau_{\text{viscous}}} \sim L \to \infty$$

$$t, \quad T = \alpha t\,, \quad T_1 = \alpha^2 t$$

ORR-SOMMERFELD EQUATION

$$\begin{cases} \mathcal{L}\psi_t - Wi\alpha^3\psi + i\alpha(-\psi\mathcal{L}W + W\mathcal{L}\psi) = \frac{1}{R_e}\mathcal{L}^2\psi \\ \square \\ \psi(r=1,t) = \frac{\partial\psi}{\partial r}(r=1,t) = 0 \\ \frac{\psi}{r}, \frac{1}{r}\frac{\partial\psi}{\partial r} \text{ bounded } \; r \to 0 \end{cases}$$

$$W = W_0 + W_1 e^{i\omega t}$$

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \alpha^2$$

$$W_0 = v(1 - r^2) \qquad W_1 = \Lambda\frac{1}{i}\left[1 - \frac{J_0(i^{3/2}\gamma r)}{J_0(i^{3/2}\gamma)}\right]$$

$\square$

$$\Lambda = \frac{R_1}{\rho\omega L_d} \qquad v = \frac{R_0}{4\mu L_d}$$

REDUCED ORR-SOMMERFELD EQUATION

$$\mathcal{L}\psi_t - Wi\alpha^3\psi + i\alpha(-\psi\mathcal{L}W + W\mathcal{L}\psi) = \frac{1}{R_e}\mathcal{L}^2\psi \quad \Rightarrow \quad \left\{ \begin{array}{l} \tilde{\mathcal{L}}\tilde{\psi}_t = \frac{1}{R_e}\tilde{\mathcal{L}}^2\tilde{\psi} \\ \tilde{\psi}(r=1,t) = \frac{\partial\tilde{\psi}}{\partial r}(r=1,t) = 0 \\ \frac{\tilde{\psi}}{r}, \frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r} \ \text{bounded} \ \ r \to 0 \end{array} \right.$$

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \alpha^2 \longrightarrow \tilde{\mathcal{L}} = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)$$

$$\tilde{\psi} = \sum_n a_n \phi_n e^{\lambda_n t}$$

$$\phi_n(r) = \frac{\sqrt{2}}{\mu_n} r\left(r - \frac{J_1(\mu_n r)}{J_1(\mu_n)}\right)$$

$$\lambda_n = -\frac{\mu_n^2}{R_e} \ \text{eigenvalues} \qquad \mu_n J_0(\mu_n) = 2J_1(\mu_n) \Longleftrightarrow J_2(\mu_n) = 0$$

$$\mathcal{F}([0,1]) = \left\{ f \in L_2([0,1]) \ : f = \frac{\partial f}{\partial r} = 0 \ \ \text{at} \ \ r = 1, \ \frac{f}{r} \ \& \ \frac{1}{r}\frac{\partial f}{\partial r} \ \text{bounded} \ \ r \to 0 \right\}$$

$$\langle f, g \rangle = -\int_0^1 \frac{1}{r} f \, \tilde{\mathcal{L}} g \, dr = \int_0^1 \frac{1}{r}\frac{\partial f}{\partial r}\frac{\partial g}{\partial r} dr$$

GALERKIN PROJECTION

$$\psi(r,t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(r) \quad \Rightarrow \quad \int_0^1 -\left(\mathcal{L}\psi_t - Wi\alpha^3\psi + i\alpha(-\psi\mathcal{L}W + W\mathcal{L}\psi) - \frac{1}{R_e}\mathcal{L}^2\psi\right)\frac{1}{r}\tilde{\mathcal{L}}\phi_k \, dr = 0$$

$$\boxed{\left(1+\frac{\alpha^2}{\gamma_n^2}\right)\frac{dA_n}{dt} - \frac{i}{\gamma_n^2}\sum_{k=1}^{\infty}[-\alpha^3 P_{nk} + \alpha H_{nk} + (-\alpha^3 Q_{nk} + \alpha G_{nk})e^{i\omega t}]A_k = \frac{1}{R_e}\left(\lambda_n R_e - 2\alpha^2 - \frac{\alpha^4}{\gamma_n^2}\right)A_n}$$

$$P_{nk} = -\int_0^1 \frac{1}{r} W_0 \tilde{\mathcal{L}}\phi_{0k}\phi_{0n} dr \qquad Q_{nk} = -\int_0^1 \frac{1}{r} W_1 \tilde{\mathcal{L}}\phi_{0k}\phi_{0n} dr$$

$$H_{nk} = -\int_0^1 \frac{1}{r}\tilde{\mathcal{L}}\phi_{0k}(-\phi_{0n}\tilde{\mathcal{L}}W_0 + W_0\tilde{\mathcal{L}}\phi_{0n})dr = -\int_0^1 \frac{1}{r} W_0 \tilde{\mathcal{L}}\phi_{0k}\tilde{\mathcal{L}}\phi_{0n} dr$$

$$G_{nk} = -\int_0^1 \frac{1}{r}\tilde{\mathcal{L}}\phi_{0k}(-\phi_{0n}\tilde{\mathcal{L}}W_1 + W_1\tilde{\mathcal{L}}\phi_{0n})dr$$

$$\frac{d\mathbf{a}}{dt} = [\mathbf{A}_0 + \mathbf{A}_1 e^{i\omega t}]\mathbf{a}$$

$$\mathbf{a}(t) = e^{\beta t}\mathbf{g}(t)$$

$\frac{dG}{dt} = [A_0 + A_1 e^{i\omega t}]G$
□
$G(T) = Q = I$

MULTI-SCALE PERTURBATION METHOD

$$A_n(t) = A_{0n}(t,T,T_1) + \alpha A_{1n}(t,T,T_1) + \alpha^2 A_{2n}(t,T,T_1) + \ldots$$

$T = \alpha t$, $T_1 = \alpha^2 t$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial T} + \alpha^2 \frac{\partial}{\partial T_1}$$

$O(1)$	$\frac{\partial A_{0n}}{\partial t} - \lambda_n A_{0n} = 0$
$O(\alpha)$	$\frac{\partial A_{1n}}{\partial t} - \lambda_n A_{1n} = S_1$
$O(\alpha^2)$	$\frac{\partial A_{2n}}{\partial t} - \lambda_n A_{2n} = S_2$

$$S_1 = -\frac{\partial A_{0n}}{\partial T} - \frac{i}{\gamma_{nn}} \sum_{k=1}^{\infty} (H_{nk} + G_{nk} e^{i\omega t}) A_{0k}$$

$$S_2 = -\frac{2}{R_e} A_{0n} + \frac{1}{\gamma_{nn}} \frac{\partial A_{0n}}{\partial t} - \frac{\partial A_{0n}}{\partial T_1} - \frac{\partial A_{1n}}{\partial T} - \frac{i}{\gamma_{nn}} \sum_{k=1}^{\infty} (H_{nk} + G_{nk} e^{i\omega t}) A_{1k}$$

$O(1)$ perturbation

$$\frac{\partial A_{0n}}{\partial t} - \lambda_n A_{0n} = 0 \quad \Longrightarrow \quad A_{0n}(t,T,T_1) = \tilde{A}_{0n}(T,T_1)e^{\lambda_n t}$$

$O(\alpha)$ perturbation

$$\frac{\partial A_{1n}}{\partial t} - \lambda_n A_{1n} = S_1$$

$$S_1 = -\underline{\left(\frac{\partial \tilde{A}_{0n}}{\partial T} + \frac{iH_{nn}}{\gamma_{nn}}\tilde{A}_{0n}\right)}e^{\lambda_n t} - \frac{i}{\gamma_{nn}}\sum_{k=1}^{\infty}\left(\left(1-\delta_{nk}\right)H_{nk} + G_{nk}e^{i\omega t}\right)\tilde{A}_{0k}e^{\lambda_k t}$$

$$\Downarrow$$

$$\frac{\partial \tilde{A}_{0n}}{\partial T} + \frac{iH_{nn}}{\gamma_{nn}}\tilde{A}_{0n} = 0 \quad \Longrightarrow \quad \tilde{A}_{0n}(T,T_1) = \hat{A}_{0n}(T_1)e^{-\frac{iH_{nn}}{\gamma_{nn}}T}$$

$$\Downarrow$$

$$A_{1n}(t,T,T_1) = \tilde{A}_{1n}(T,T_1)e^{\lambda_n t} - \frac{i}{\gamma_{nn}}\sum_{k=1}^{\infty}\left(\left(1-\delta_{nk}\right)H_{nk}\,\frac{e^{\lambda_n t}-e^{\lambda_k t}}{\lambda_n-\lambda_k} + G_{nk}\,\frac{e^{\lambda_n t}-e^{(\lambda_k+i\omega)t}}{\lambda_n-(\lambda_k+i\omega)}\right)\hat{A}_{0k}e^{-\frac{iH_{kk}}{\gamma_{kk}}T}$$

$O(\alpha^2)$ perturbation

$$S_2 = -\frac{2}{R_e}A_{0n} + \frac{1}{\gamma_{nn}}\frac{\partial A_{0n}}{\partial t} - \frac{\partial A_{0n}}{\partial T_1} - \frac{\partial A_{1n}}{\partial T} - \frac{i}{\gamma_{nn}}\sum_{k=1}^{\infty}(H_{nk} + G_{nk}e^{ik\omega t})A_{1k}$$

$$\frac{\partial \bar{A}_{1n}}{\partial T} + \frac{iH_{nn}}{\gamma_{nn}}\bar{A}_{1n} = S_T$$

$$S_T = -\frac{2}{R_e}\hat{A}_{0n}e^{-\frac{iH_{nn}}{\gamma_{nn}}T} + \frac{\hat{A}_{0n}\lambda_n}{\gamma_{nn}}e^{-\frac{iH_{nn}}{\gamma_{nn}}T} - \frac{d\hat{A}_{0n}}{dT_1}e^{-\frac{iH_{nn}}{\gamma_{nn}}T} +$$

□

$$+\sum_{k=1}^{\infty}\left[H_{nk}\frac{1-\delta_{nk}}{\lambda_n-\lambda_k} + \frac{G_{nk}}{\lambda_n-(\lambda_k+ik\omega)}\right]\left(\frac{H_{kk}}{\gamma_{kk}\gamma_{nn}} - \frac{H_{nn}}{\gamma_{nn}^2}\right)\hat{A}_{0k}e^{-\frac{iH_{kk}}{\gamma_{kk}}T} +$$

□

$$+\frac{\hat{A}_{0n}e^{-\frac{iH_{nn}}{\gamma_{nn}}T}}{\gamma_{nn}}\sum_{k=n}^{\infty}\frac{1}{\gamma_{kk}}\frac{H_{nk}H_{kn}}{\lambda_k-\lambda_n}$$

$$\frac{d\hat{A}_{0n}}{dT_1} + \sigma_n\hat{A}_{0n} = 0 \longrightarrow \hat{A}_{0n}(T_1) = a_{0n}e^{-\sigma_n T_1}$$

$$\bar{A}_{1n}(T,T_1) = \hat{A}_{1n}(T_1)e^{-\frac{iH_{nn}}{\gamma_{nn}}T} + \sum_{k=1}^{\infty}(1-\delta_{nk})\left(\frac{H_{nk}}{\lambda_n-\lambda_k} + \frac{G_{nk}}{\lambda_n-(\lambda_k+ik\omega)}\right)\left(\frac{H_{kk}}{\gamma_{kk}\gamma_{nn}} - \frac{H_{nn}}{\gamma_{nn}^2}\right)\hat{A}_{0k}\frac{e^{-\frac{iH_{nn}}{\gamma_{nn}}T} - e^{-\frac{iH_{kk}}{\gamma_{kk}}T}}{-\frac{iH_{nn}}{\gamma_{nn}} + \frac{iH_{kk}}{\gamma_{kk}}}$$

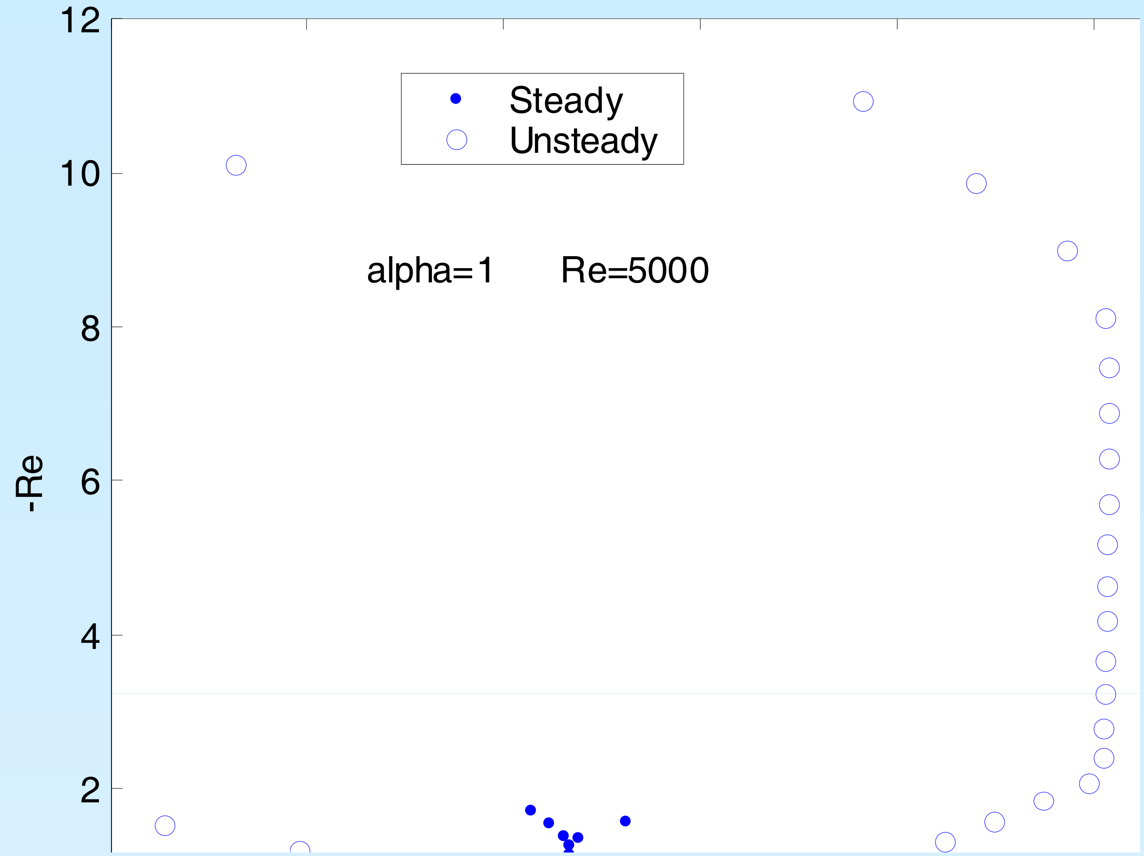
NUMERICAL RESULTS

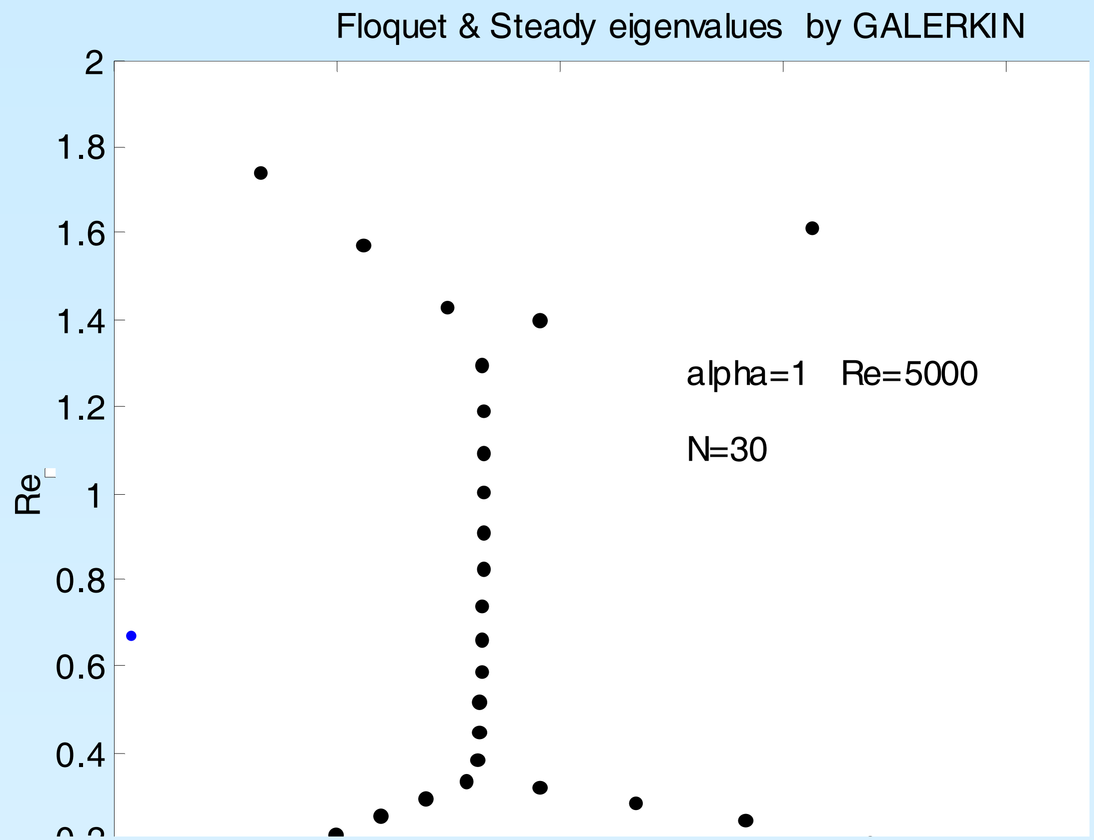
alpha=0.1 Re=500 N=30

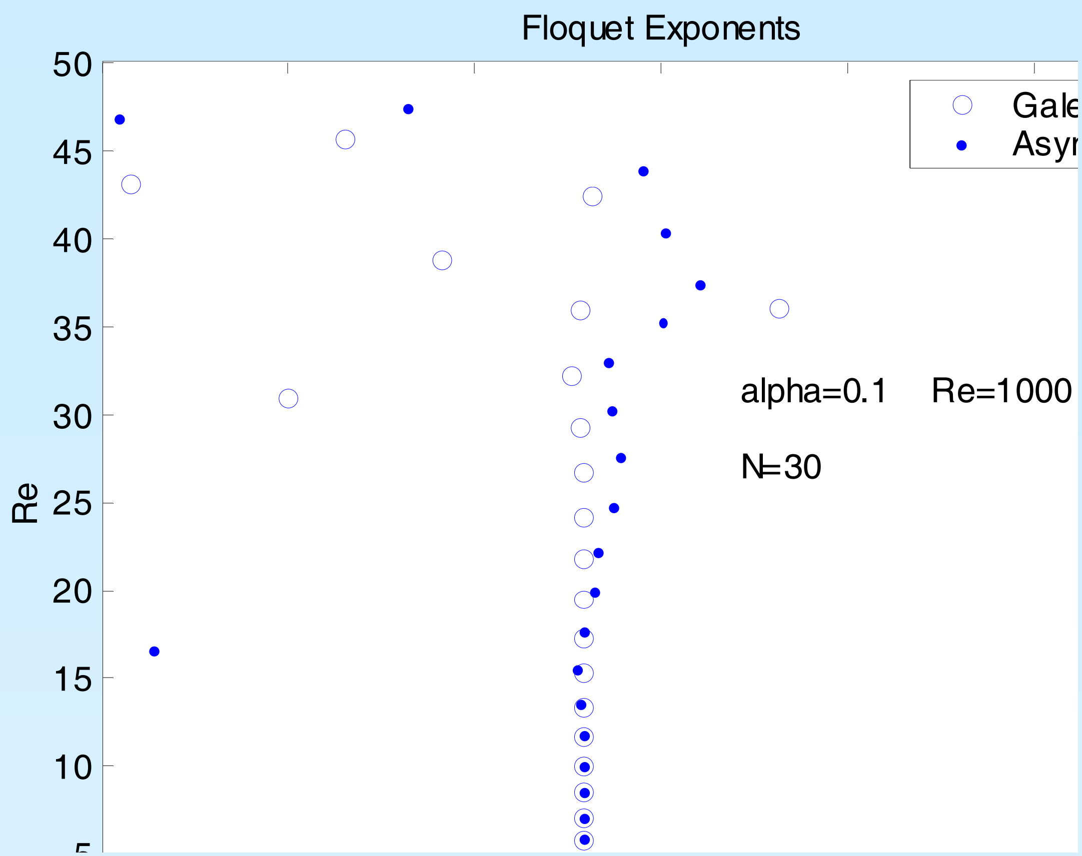
Galerkin	Asymptotic
-0.3606 - 0.4206i	-0.3605 - 0.4189i
-0.8827 - 0.4170i	-0.8828 - 0.4189i
-1.6920 - 0.4189i	-1.6915 - 0.4189i
-2.7481 - 0.4189i	-2.7475 - 0.4189i
-4.0513 - 0.4189i	-4.0507 - 0.4189i
-5.6023 - 0.4189i	-5.6017 - 0.4188i
-7.4010 - 0.4189i	-7.4008 - 0.4186i
-9.4477 - 0.4189i	-9.4497 - 0.4183i
-11.7424 - 0.4189i	-11.7569 - 0.4205i
-14.2850 - 0.4189i	-14.3784 - 0.4507i

alpha=0.1 Re=5000 N=60

Galerkin	Asymptotic
-0.1595 - 0.2335i	-0.0238 - 0.4192i
-0.2579 - 0.4201i	-0.1212 - 0.4184i
-0.1854 - 0.4664i	-0.2434 - 0.4188i
-0.0958 - 0.5478i	-0.3147 - 0.4190i
-0.3857 - 0.4219i	-0.3831 - 0.4189i
-0.5455 - 0.4206i	-0.5441 - 0.4189i
-0.7287 - 0.4199i	-0.7277 - 0.4189i
-0.9357 - 0.4195i	-0.9349 - 0.4189i
-1.1669 - 0.4193i	-1.1662 - 0.4189i
-1.4225 - 0.4192i	-1.4218 - 0.4189i


12
10
8
6
4
2
-Re
Steady
Unsteady
alpha=1 Re=5000


Floquet & Steady eigenvalues by GALERKIN
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
Re
alpha=1 Re=5000
N=30



Floquet Exponents
Re
50
45
40
35
30
25
20
15
10
Gale
Asyr
alpha=0.1 Re=1000
N=30