SPECTRAL ANALYSIS IN NUMERICAL METHODS

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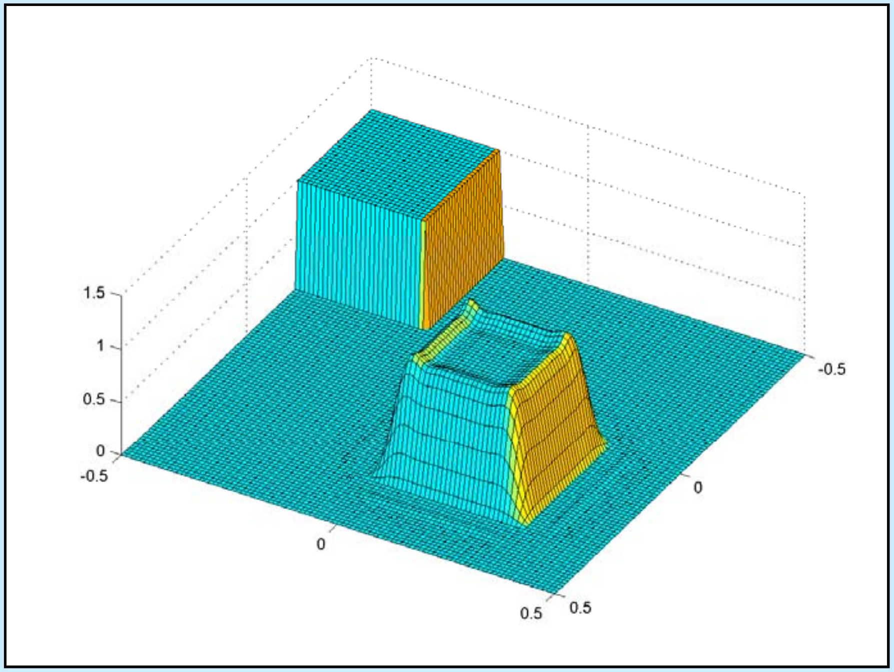

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ACCURACY OF A NUMERICAL SCHEME

Order of convergence in sup-norm

Valid for initial smooth conditions

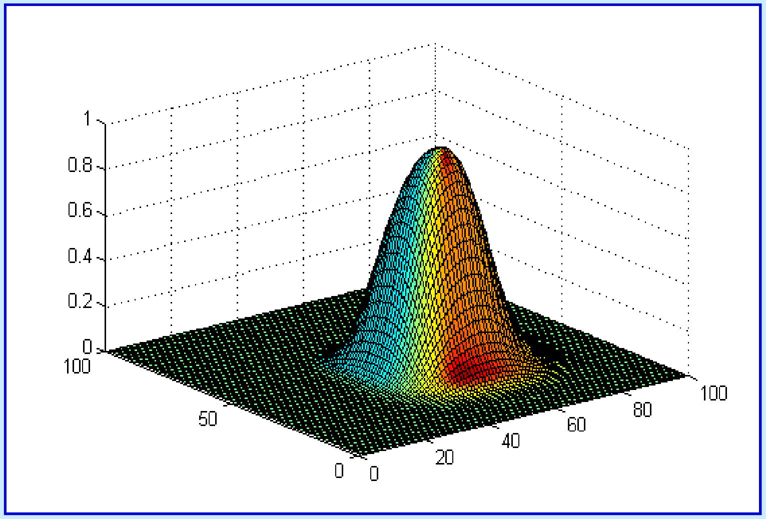
for sharp front initial condition

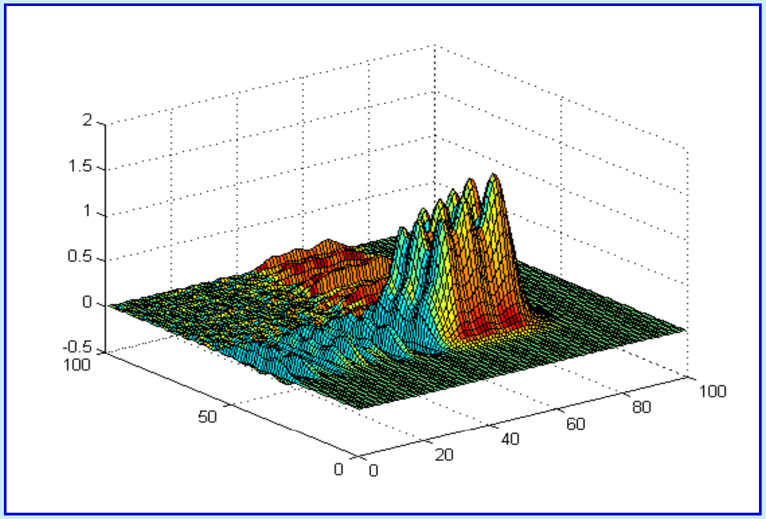
*higher order schemes give "**oscillatory behavior**"*

*lower order schemes (up-wind)   "**may work better**"*

WHY ???

SIMULATIONS



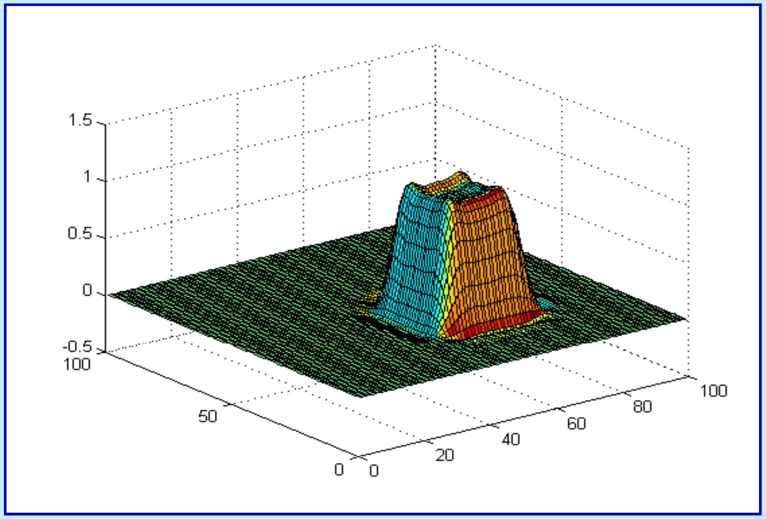


FDM up-wind

TRANSLATING TOWER

FDM no up-wind

LOCOM



SPECTRAL ANALYSIS

Analysis of the kernels
of a *continuous operator*
and its *discrete equivalent*

- Semi-discrete operator (only spatial)

- Discrete operator (spatial+time)

THE PURE ADVECTIVE OPERATOR

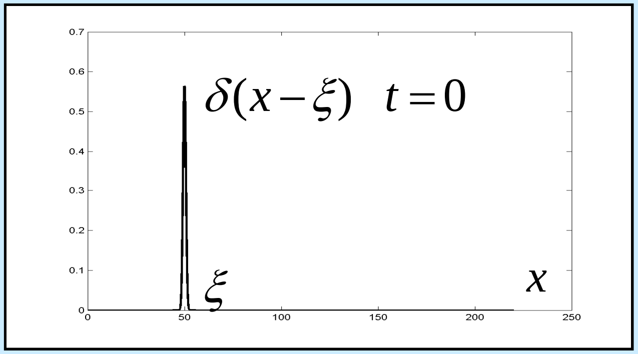
$$\begin{cases} \mathcal{L}(u) = 0 \\ u(x, t=0) = u_0(x) \\ u(0,t) = 0 \quad u(l,t) = 0 \end{cases} \qquad \mathcal{L} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$$

Time domain

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi, t) u_0(\xi) d\xi$$

$$\begin{cases} \mathcal{L}(G) = 0 \\ u(x, t=0) = \delta(x-\xi) \\ u(0,t) = 0 \quad u(l,t) = 0 \end{cases}$$

G(x,t) impulse response



Initial conditions

SEMI-DISCRETE SPATIAL OPERATOR

Continuous pure advective operator

$$\mathcal{L}(u) = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$$

Spatial discretization (Galerkin , Ritz , FDM)

Semi-discrete pure advective operator

$$\mathcal{L}_x(u)_i = a_1\frac{du_{i-1}}{dt} + a_2\frac{du_i}{dt} + a_3\frac{du_{i+1}}{dt} + b_1 u_{i-1} + b_2 u_i + b_3 u_{i+1}$$

IMPULSE RESPONSE - SOLUTION

Analytical solution for initial condition $u_n(t=0)=\delta_{0,n} \quad \forall\, n \in Z$

$$a_1 \frac{du_{i-1}}{dt} + a_2 \frac{du_i}{dt} + a_3 \frac{du_{i+1}}{dt} + b_1 u_{i-1} + b_2 u_i + b_3 u_{i+1} = 0 \qquad \forall\, i = 1,2,..,N_X$$

$$G(z,t) = \sum_{n=-\infty}^{\infty} u_n(t) z^n$$

Generating function

If G analytic $\Longrightarrow$ $$u_n(t) = \frac{1}{2\pi i} \oint_{\Gamma} G(z,t)\; z^{-n}\, dz$$

$$G(z,t) = \exp\left[-\frac{b_1 z^2 + b_2 z + b_3}{a_1 z^2 + a_2 z + a_3} t \right]$$

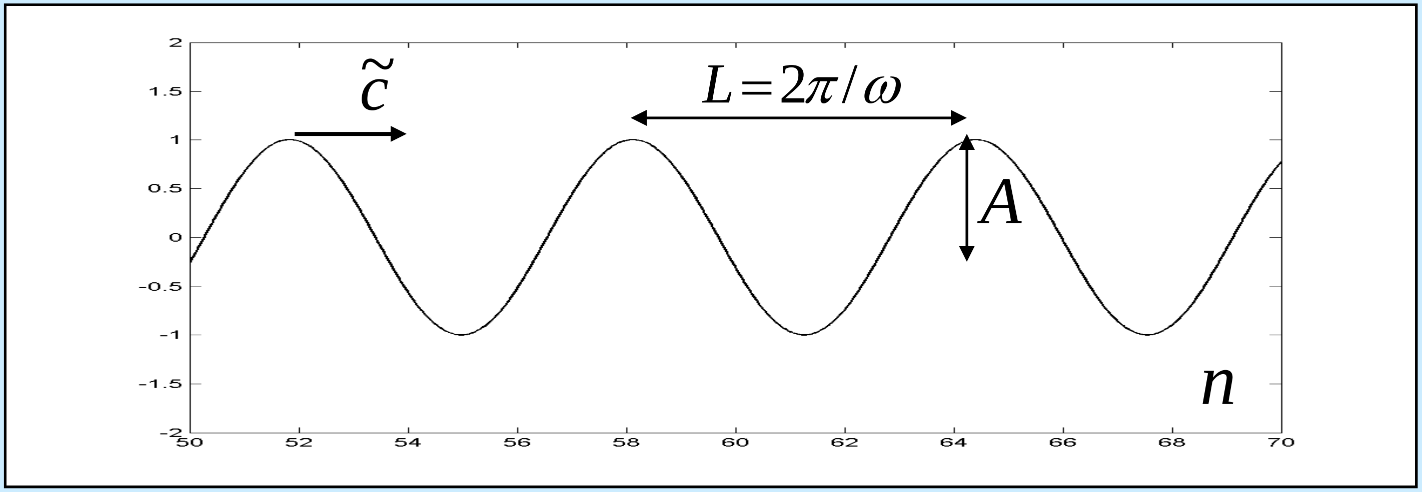
IMPULSE RESPONSE - WAVE VELOCITY

Analytical solution for initial condition $h_n(t=0)=\delta_{0,n} \quad \forall n \in Z$

$$h_n(t)=\frac{1}{\pi}\int_0^{\pi} e^{-R(\omega)t}\cos\left\{\omega\left[\frac{P(\omega)}{\omega}t-n\right]\right\}d\omega$$

Wave amplitude (A)

Wave velocity $\tilde{c}=\frac{P(\omega)}{\omega}$



IMPULSE RESPONSE - GROUP VELOCITY

$\cong 1$ **undamped scheme or short time behavior**

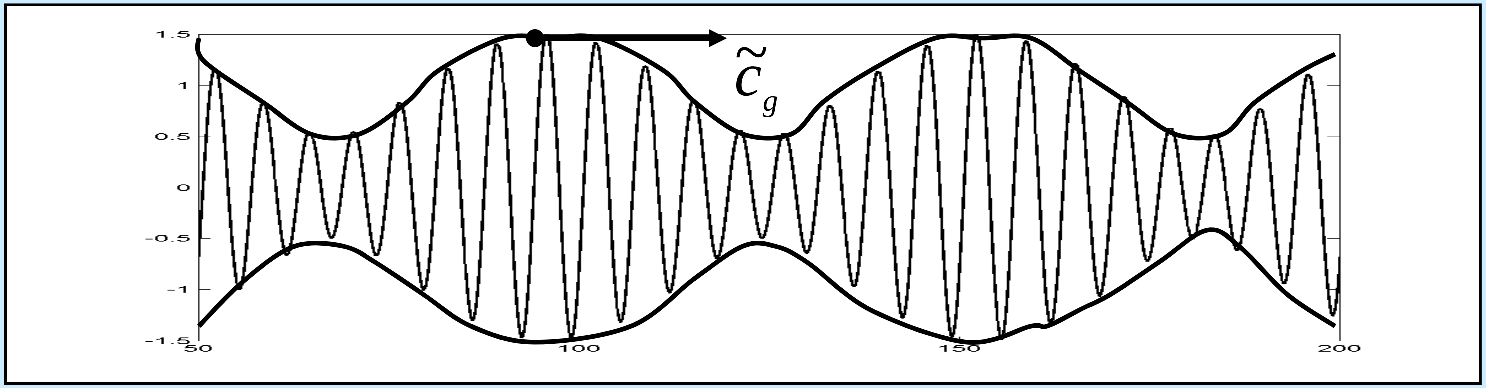
$$h_n(t) = \frac{1}{\pi}\int_0^{\pi} e^{-R(\omega)t}\cos[P(\omega)t - n\omega]\,d\omega$$

Integration by parts

$$h_n(t) = \frac{1}{\pi}\int_0^{\pi} \omega\left[\frac{dP}{d\omega}t - n\right]\sin[P(\omega)t - n\omega]\,d\omega$$

Wave envelope

$$\tilde{c}_g = \frac{d\,P}{d\omega}$$ **Group velocity**



IMPULSE RESPONSES

Central difference FDM

$$\hat{\mathcal{L}}(u) = \frac{du_i}{dt} - \frac{1}{2}u_{i-1} + \frac{1}{2}u_{i+1}$$

$$h_n(t) = J_n(t)$$

Up-wind FDM

$$\hat{\mathcal{L}}(u) = \frac{du_i}{dt} - u_{i-1} + u_i$$

$$h_n(t) = e^{-t}\frac{t^n}{n!}$$

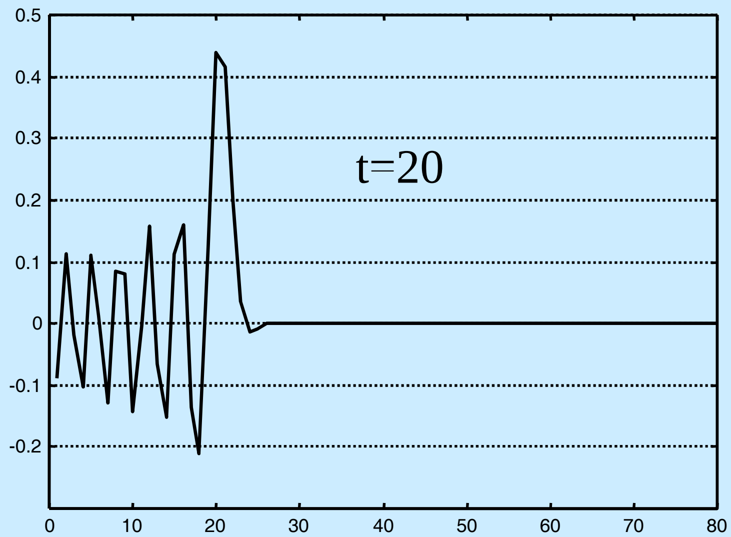
FEM

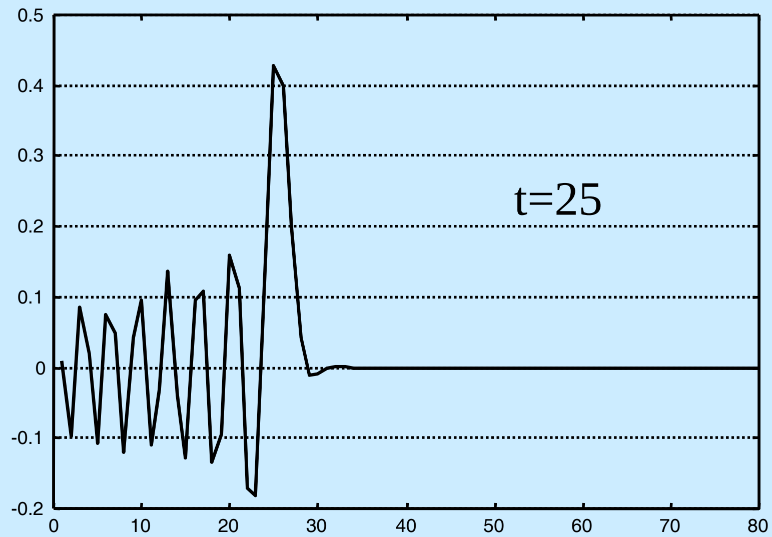
$$\hat{\mathcal{L}}(u) = \frac{1}{6}\frac{du_{i-1}}{dt} + \frac{2}{3}\frac{du_i}{dt} + \frac{1}{6}\frac{du_{i+1}}{dt} - \frac{1}{2}u_{i-1} + \frac{1}{2}u_{i+1}$$

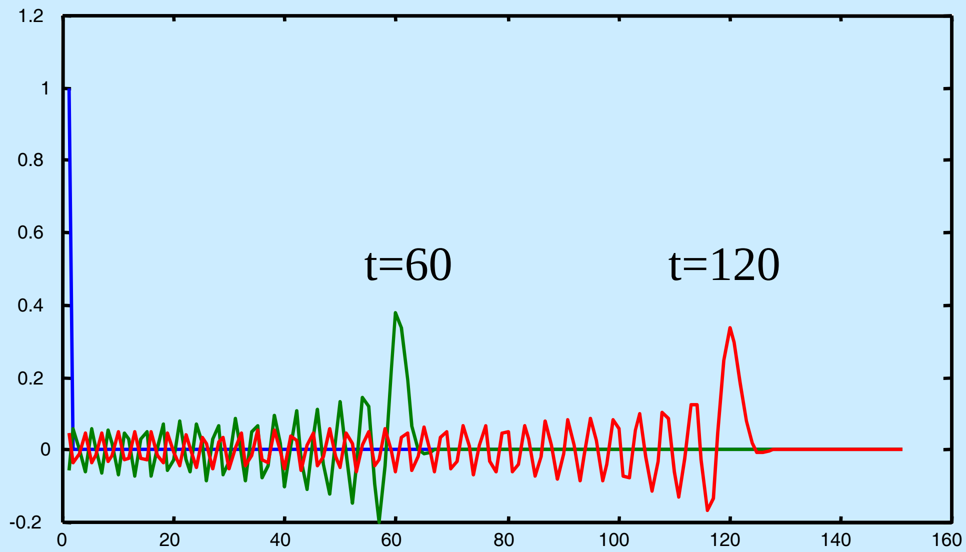
$$h_n(t) = \frac{1}{\pi}\int_0^{\pi}\cos\left[\frac{2+\cos\omega}{6\sin\omega}t - n\omega\right]d\omega$$

$$h_n(t) = \frac{1}{\pi}\int_0^{\pi} e^{-R(\omega)t}\cos\left[P(\omega)t - n\omega\right]d\omega$$

IMPULSE RESPONSE - FEM

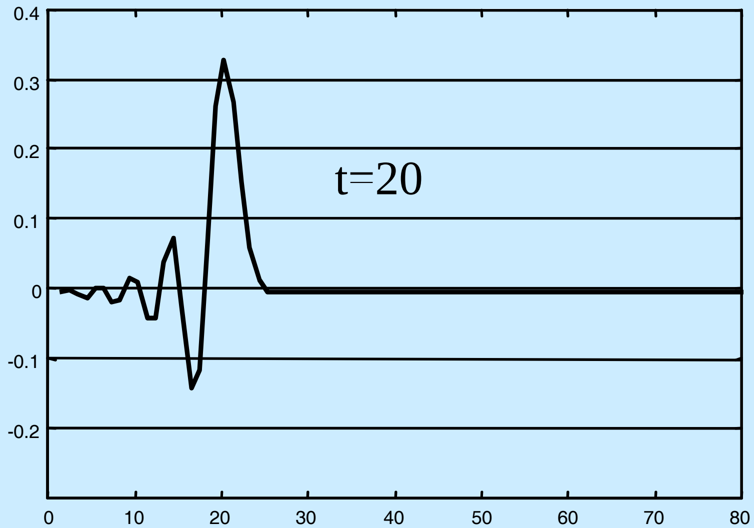


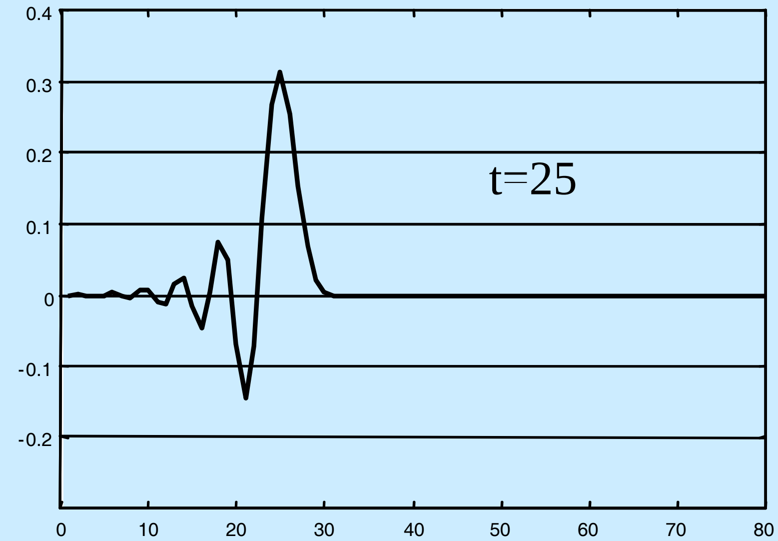




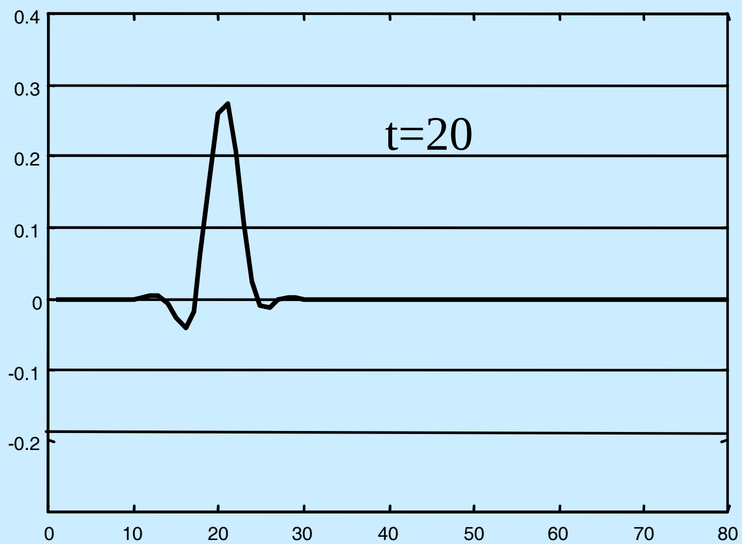
IMPULSE RESPONSE - LOCOM

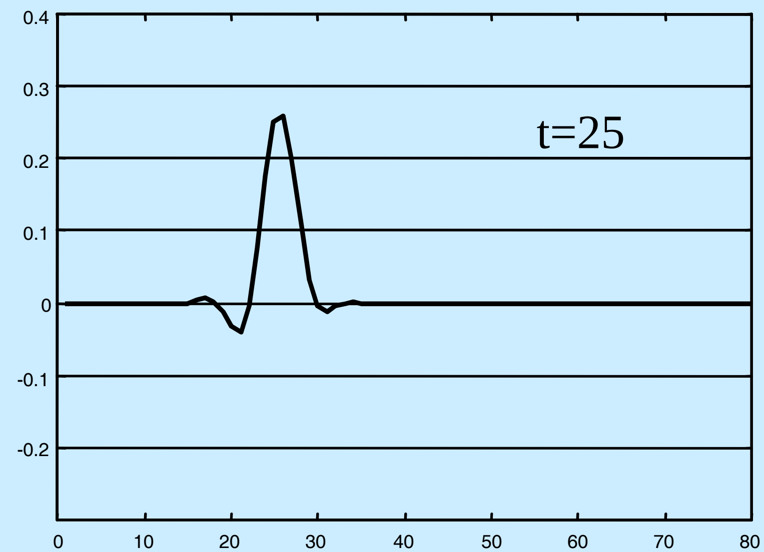
Partial up-wind



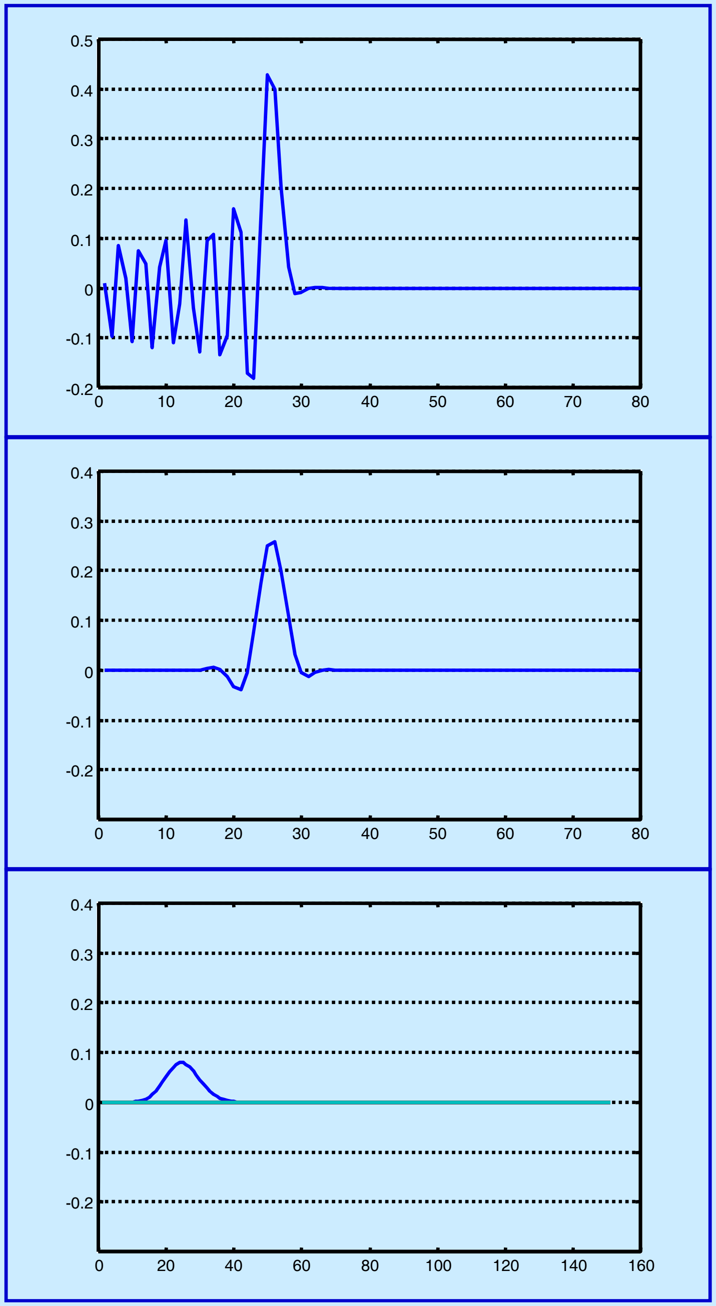


Full up-wind





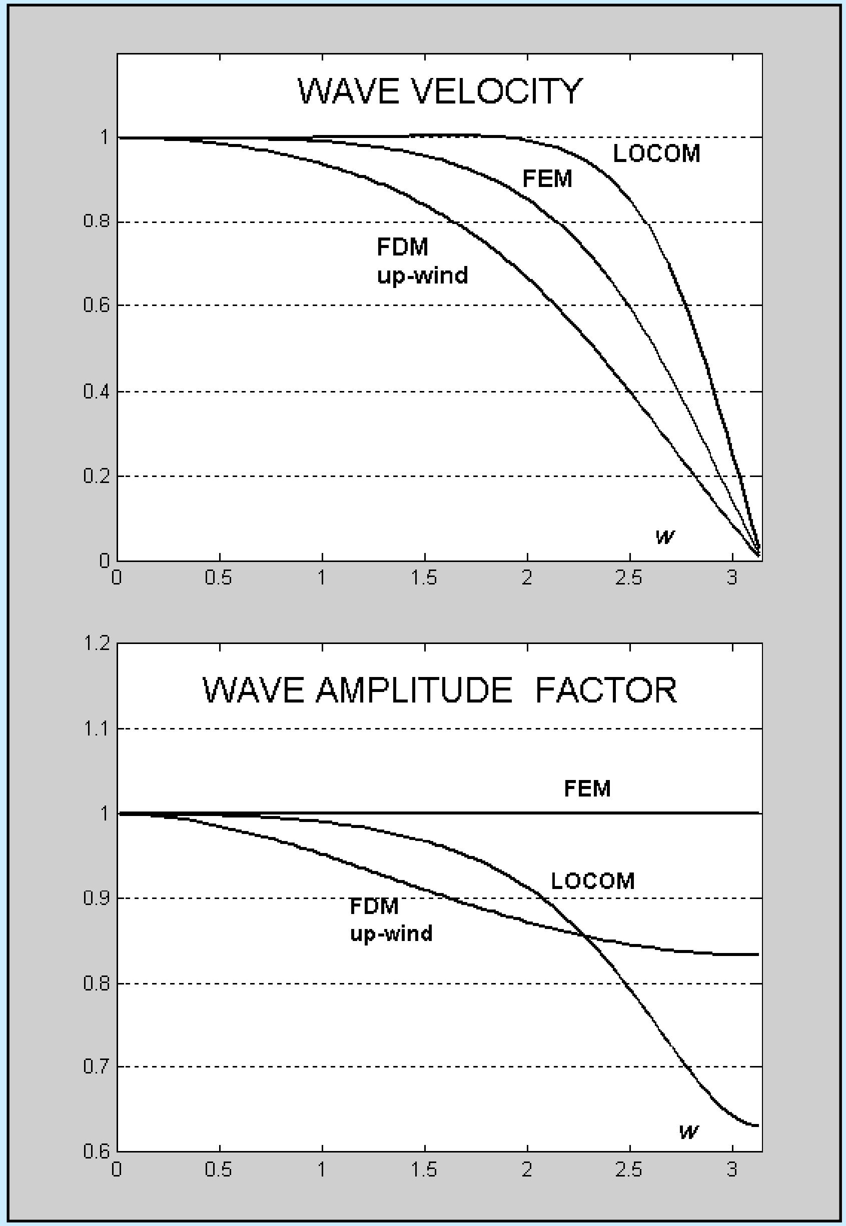
$$h_n(t) = \frac{1}{\pi}\int_0^{\pi} e^{-R(\omega)t}\cos\left[P(\omega)t - n\omega\right]d\omega$$

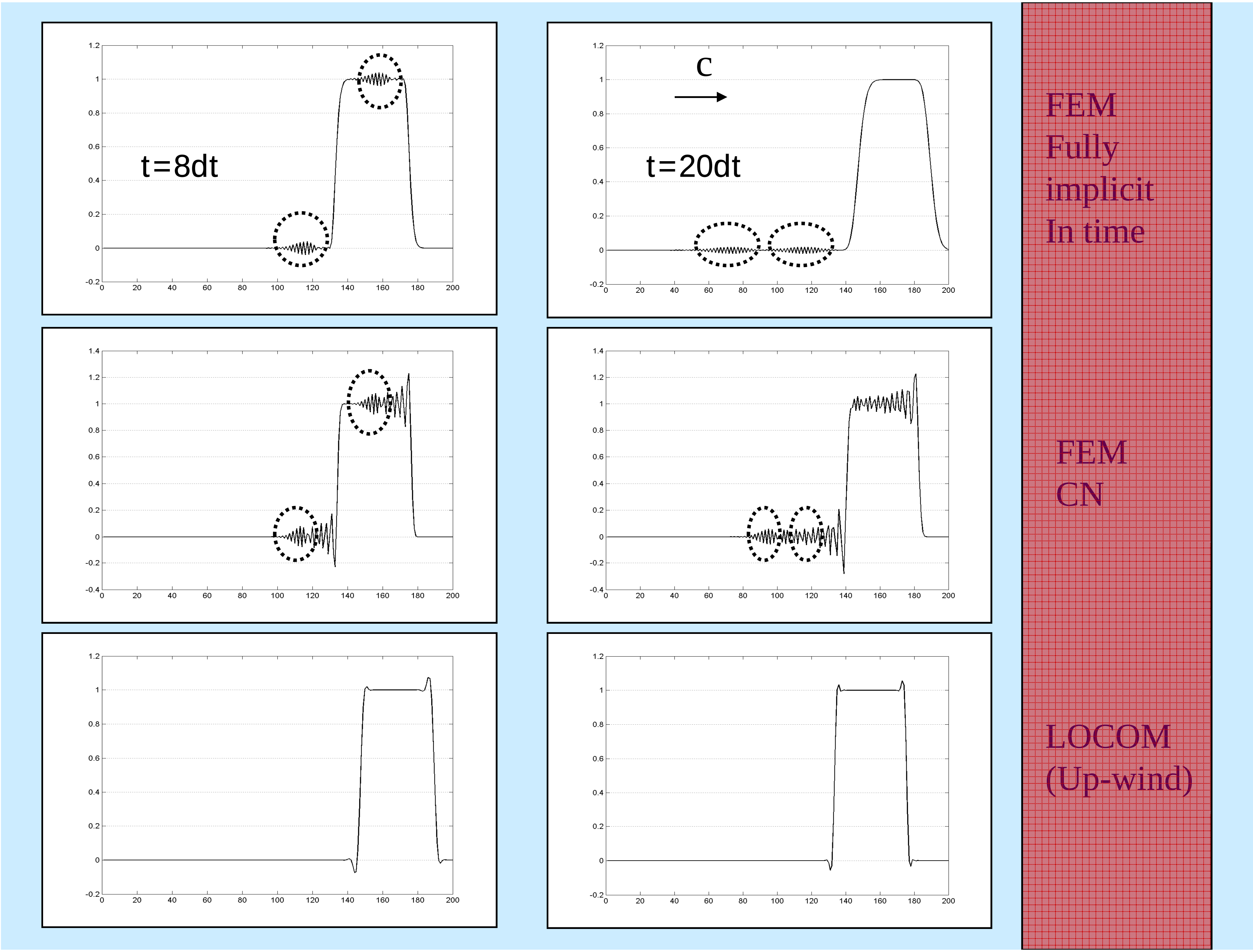


FEM

LOCOM

FDM

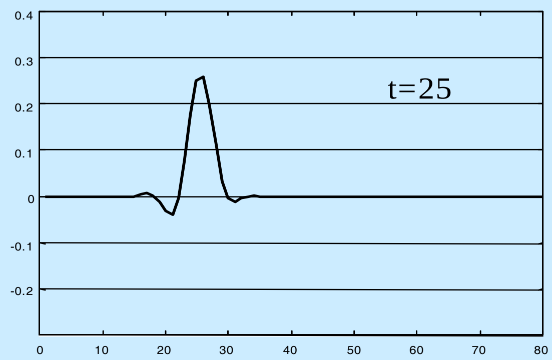



t=8dt
c
t=20dt
FEM
Fully
implicit
In time
FEM
CN
LOCOM
(Up-wind)

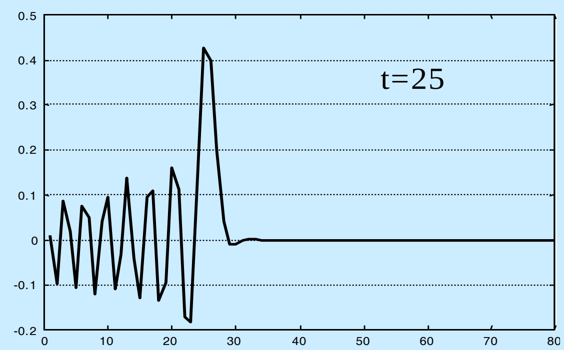
$$h_n(t) = \frac{1}{\pi}\int_0^{\alpha} \omega\left[\frac{dP}{d\omega}t - n\right]\sin\left[P(\omega)t - n\omega\right]d\omega + \frac{1}{\pi}\int_{\alpha}^{\pi} \omega\left[\frac{dP}{d\omega}t - n\right]\sin\left[P(\omega)t - n\omega\right]d\omega$$

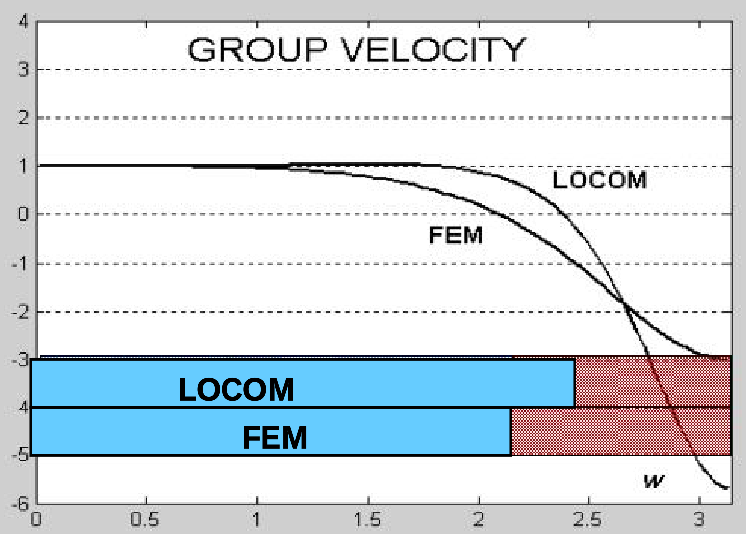
Principal Group | Secondary group

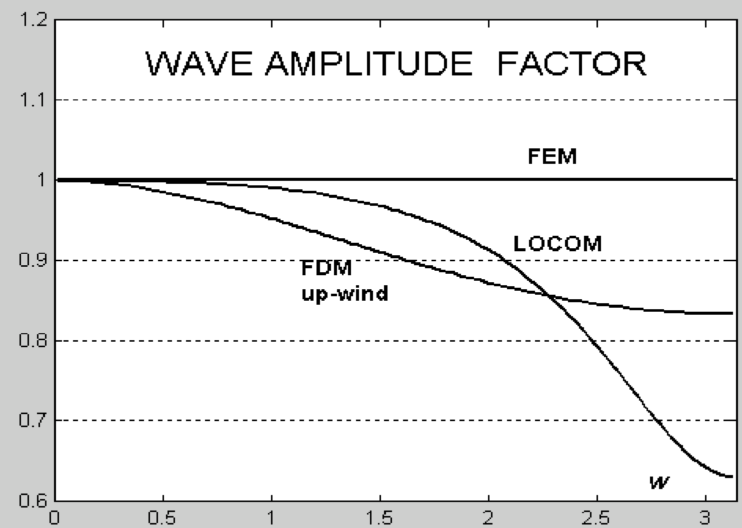
LOCOM (up-wind)



FEM (no up-wind)







CONCLUSIONS

-Spectral analysis helps to understand the intrinsic properties of a continuous operator and its discrete equivalent

-the impulse response helps to understand how an initial disturbance in space is propagated by the operator

-wave velocity and group velocity spectrums intrinsic properties of a numerical scheme

- two levels of analysis : semi-discrete operator (spatial)
fully discrete operator (spatial +time)