FREAK WAVES IN RANDOM SEAS AND STREAKS IN CHANNEL FLOWS : POSSIBLE STOCHASTIC SIMILARITIES

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Outline

- EXTREME EVENTS IN GAUSSIAN SEAS
  - LINEAR COHERENT STRUCTURE : THE WAVE GROUP
- ***WEAKLY*** NONLINEAR EFFECTS
  - NONLINEAR EVOLUTION OF THE LINEAR WAVE GROUP
    - INTERMITTENCY AND INITIAL STAGE OF FREAK WAVES
- ***STRONG*** NONLINEAR EFFECTS
  - QUASI-SOLITONIC TURBULENCE (FREAK WAVES)
- FUTURE RESEARCH:
  - ARE STREAKS SIMILAR TO FREAK WAVES ?

Freak waves, rogue waves and giant waves

Nonlinear water waves

Gaussian seas
and extreme waves

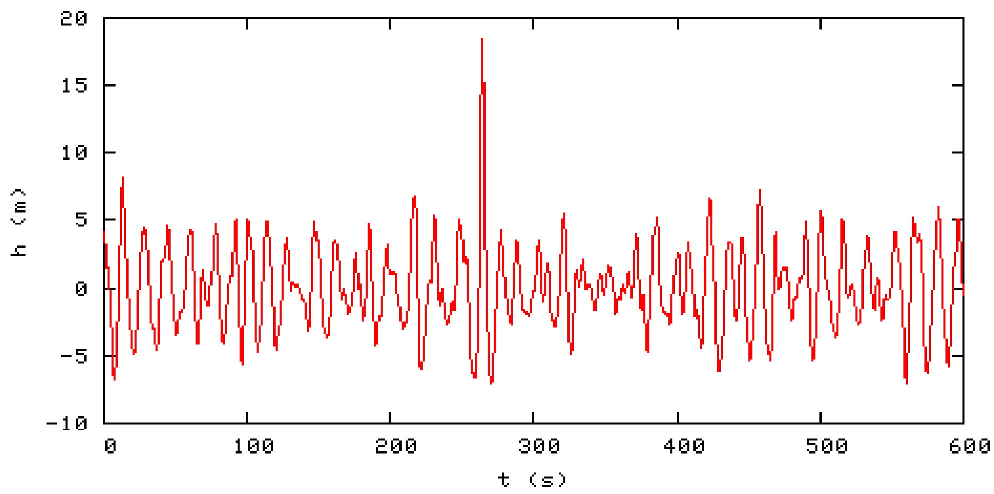
Freak waves

Rogue waves

Giant waves

Extreme waves

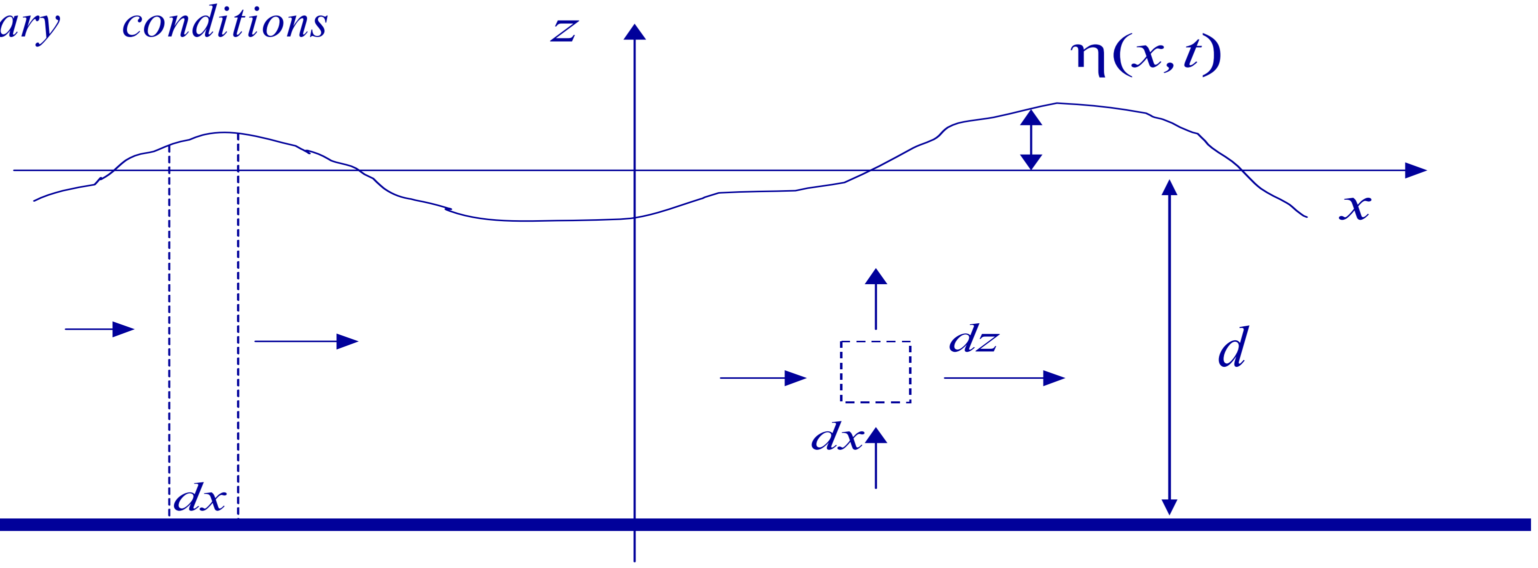
DRAUPNER EVENT JANUARY 1995



Stokes Equations for regular waves

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial z^2} = 0 \\[2ex] \left(\dfrac{\partial \Phi}{\partial z}\right)_{z=\eta} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \eta}{\partial x}\left(\dfrac{\partial \Phi}{\partial x}\right)_{z=\eta} \\[2ex] \left(\dfrac{\partial \Phi}{\partial t}\right)_{z=\eta} + \dfrac{1}{2}\left[\left(\dfrac{\partial \Phi}{\partial x}\right)^2 + \left(\dfrac{\partial \Phi}{\partial z}\right)^2\right]_{z=\eta} + g\eta = f(t) \\[2ex] boundary \quad conditions \end{cases}$$

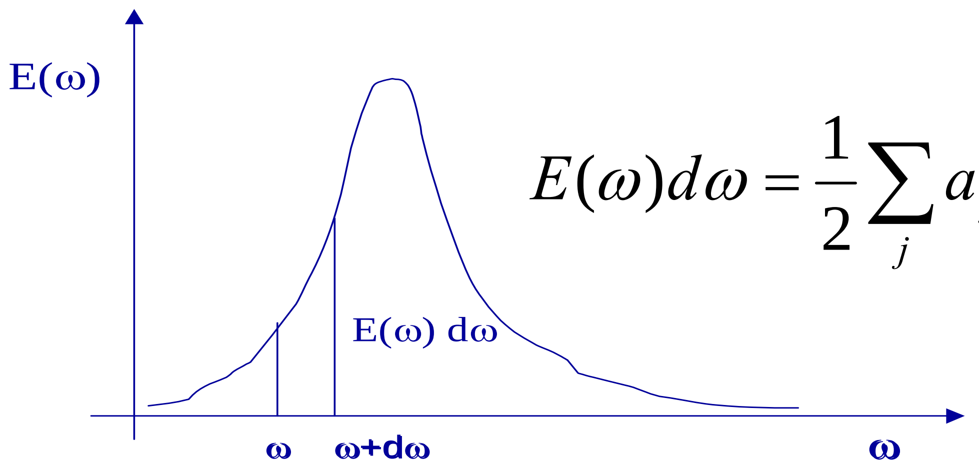
$$v_z = \frac{\partial \Phi}{\partial z} \qquad v_x = \frac{\partial \Phi}{\partial x}$$



Gaussian seas

$$\eta(x,t) = \sum_{j=1}^{N} a_j \cos\left(k_j x + \omega_j t + \varepsilon_j\right)$$

$$\varepsilon_j \quad \textit{uniformly random in } [0,2\pi]$$



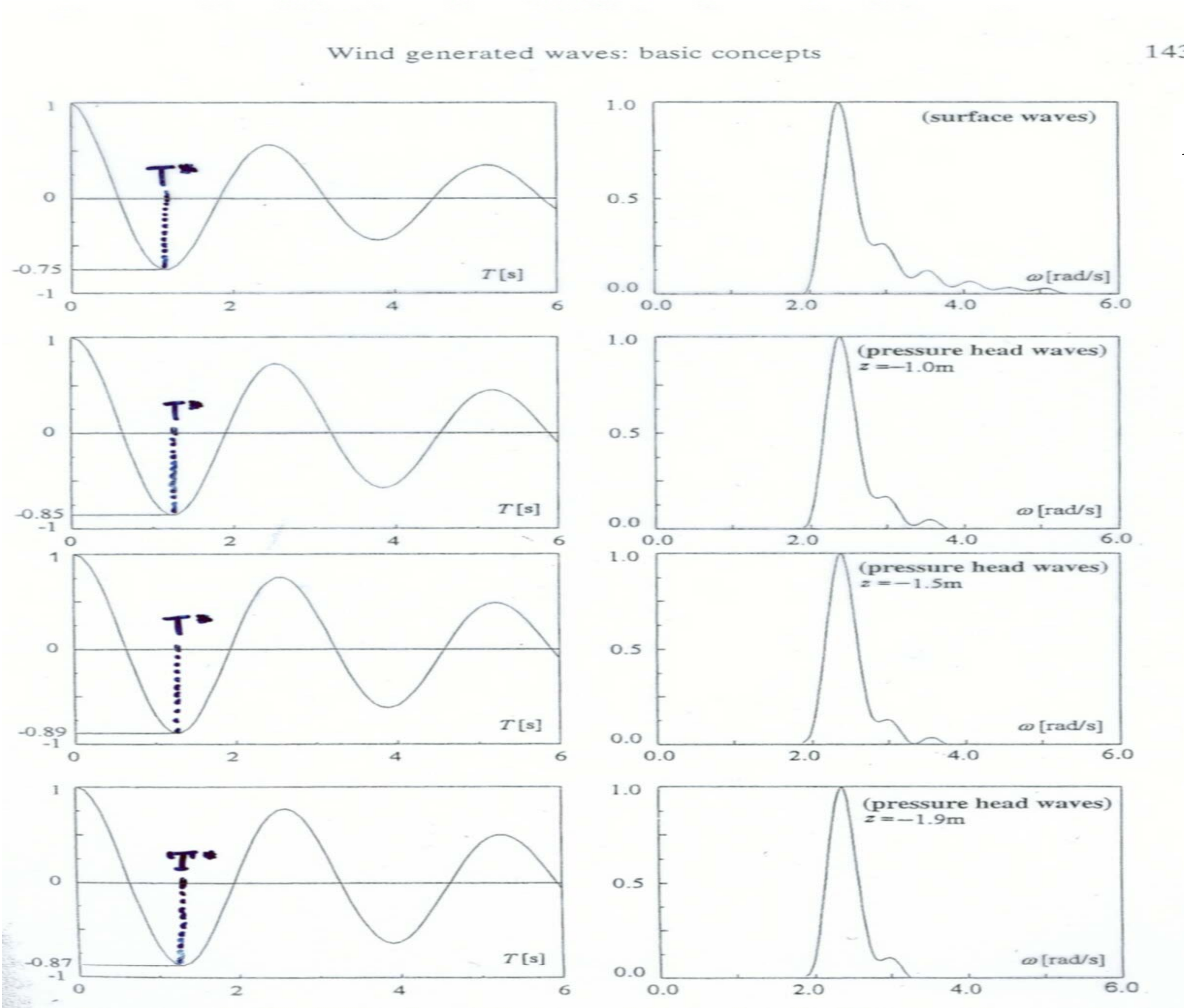
$$E(\omega)d\omega = \frac{1}{2}\sum_j a_j^2 \qquad \omega < \omega_j < \omega + d\omega$$

$$\psi(T) = \langle \eta(x_0,t_0)\eta(x_0,t_0+T) \rangle = \int_0^\infty E(\omega)\cos\omega T\, d\omega$$

Stationarity Ergodicity Gaussianity

Typical wave spectra from Mediterranean sea*

$\psi(T)$ $E(\omega)$

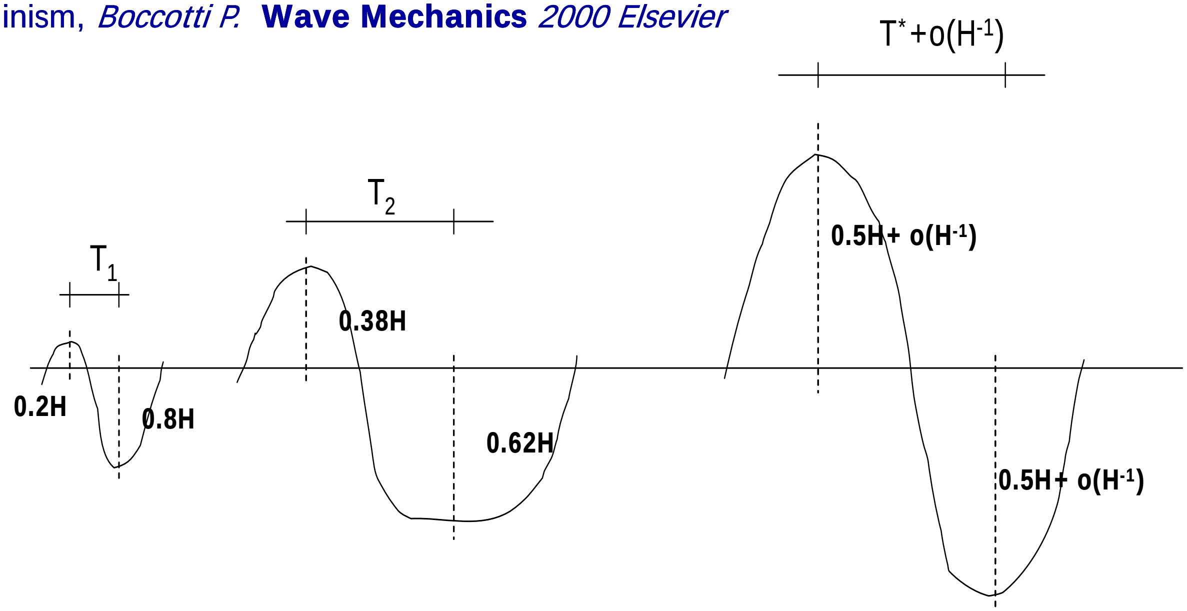


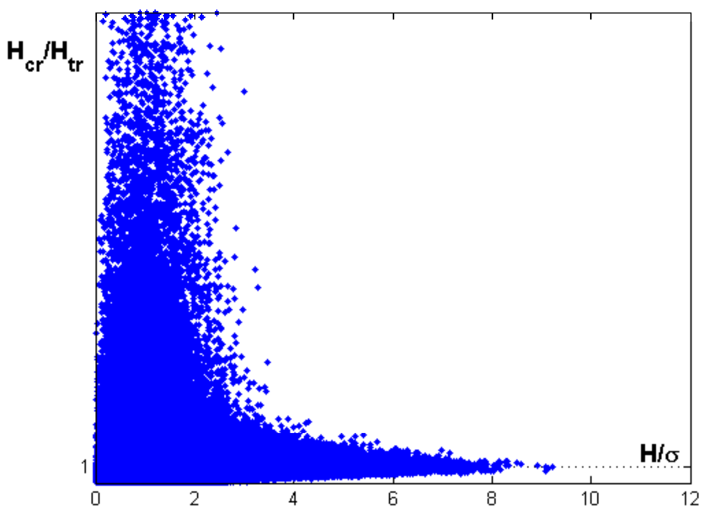
*from *Boccotti P.* **Wave Mechanics** *2000 Elsevier*

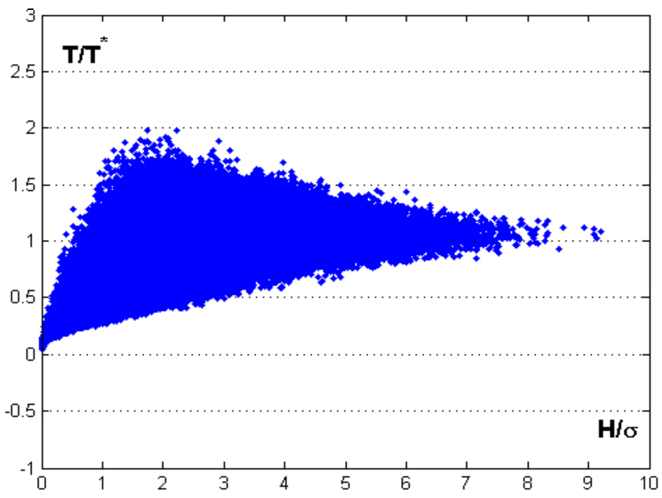
Time domain analysis

NECESSARY AND SUFFICIENT CONDITIONS TO HAVE A HIGH WAVE

*Theory of quasi-determinism, *Boccotti P.* **Wave Mechanics** *2000 Elsevier*







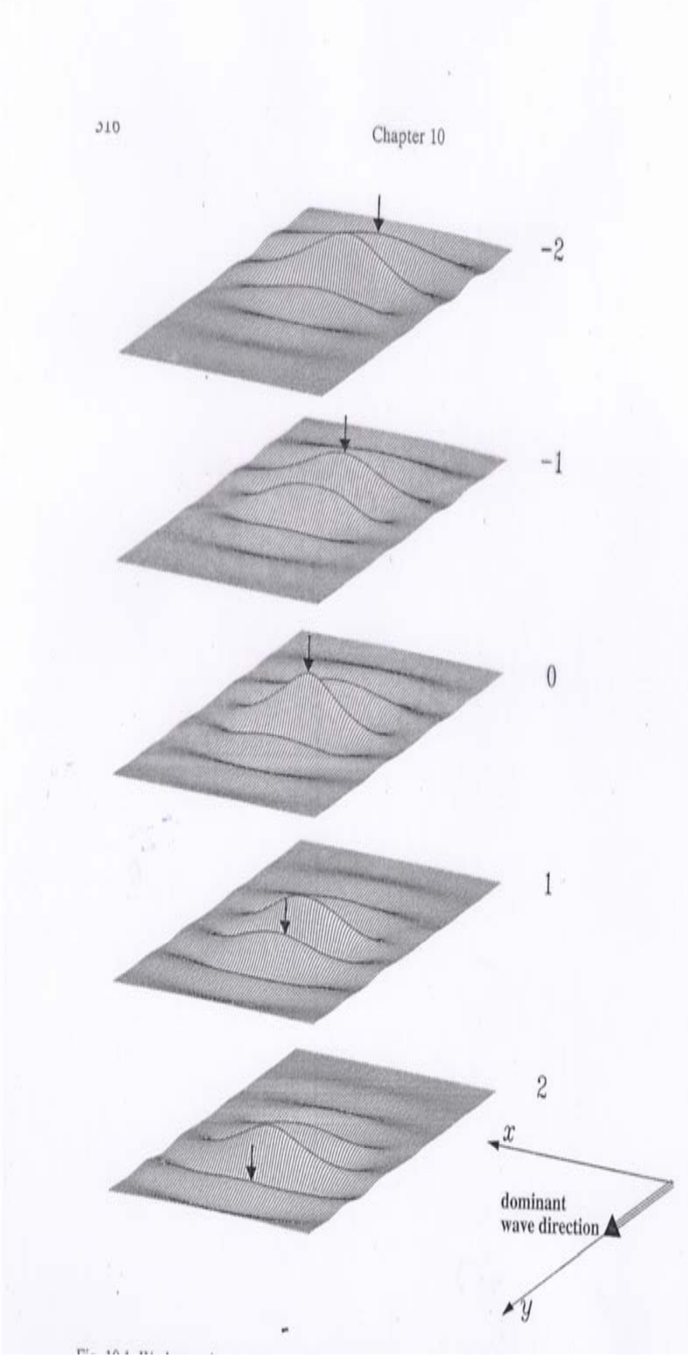
Space-time domain analysis*

What happens in the neighborhood of a point $\mathbf{x}_0$ if a large crest followed by large trough are recorded in time at $\mathbf{x}_0$?

$$\Pr\left[\begin{array}{l} \eta(\mathrm{x}_0+\mathrm{X}, t_0+T) \in (u, u+du) \\ \\ conditioned \quad to \\ \eta(\mathrm{x}_0, t_0) = H/2, \;\; \eta(\mathrm{x}_0, t_0+T_2^*) = -H/2 \end{array}\right]$$

$$\Downarrow \qquad \frac{H}{\sigma} \to \infty$$

$$\delta\left(u - \eta_c\left(\mathbf{x}_0 + \mathbf{X}, t_0 + T\right)\right)$$

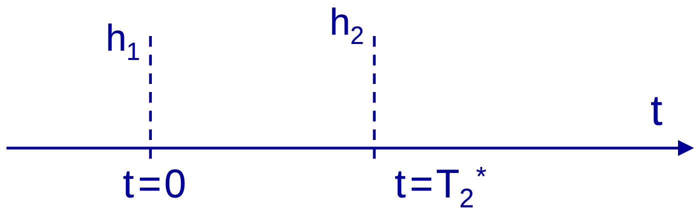


* *Boccotti P.* **Wave Mechanics** *2000 Elsevier*

Time domain analysis: successive wave crests*

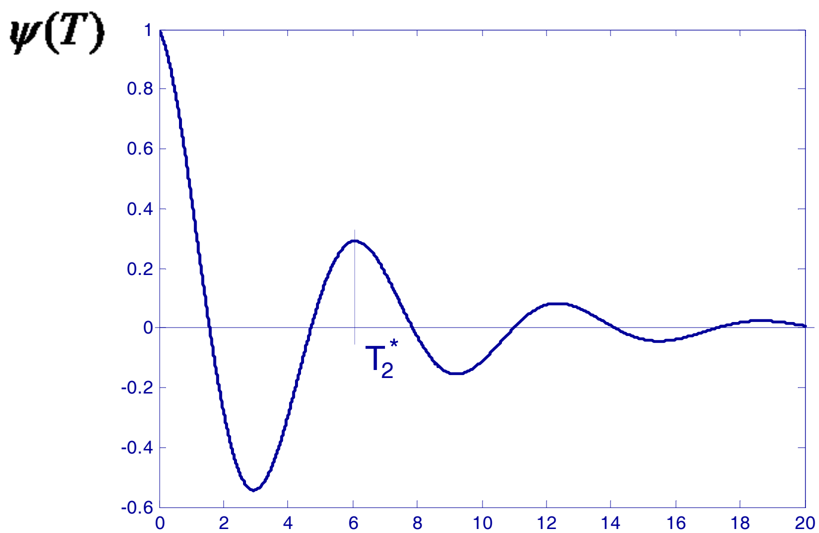
Necessary and sufficient conditions for the occurrence of two high wave crests

$$\eta(t_0) = h_1, \quad \eta(t_0 + T_2^*) = h_2$$

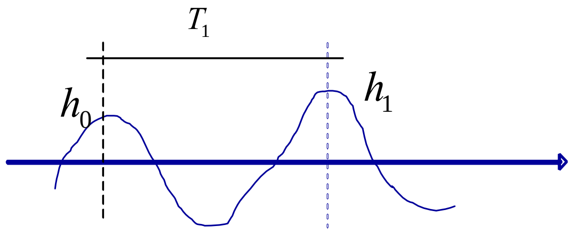


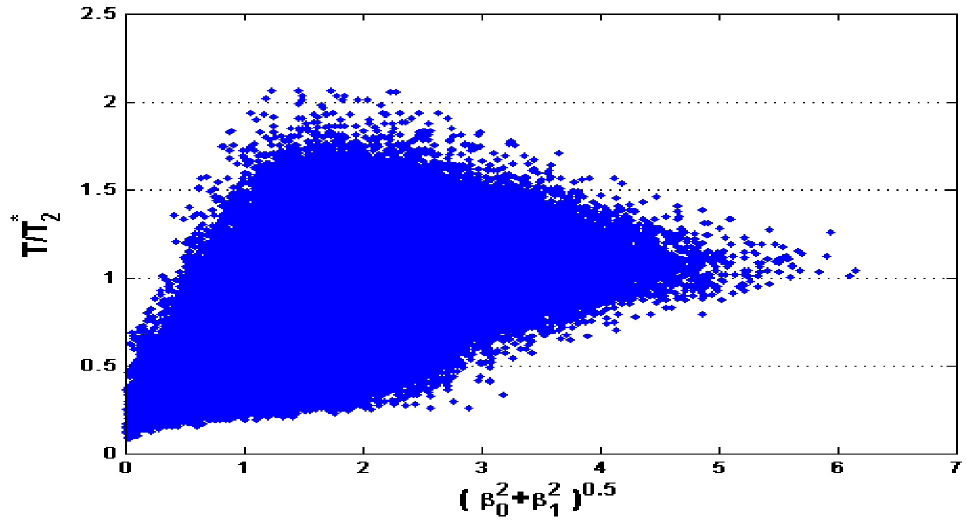
Autocovariance function

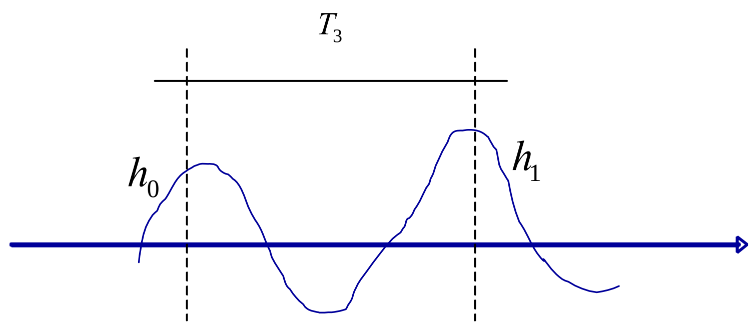
$$\psi(T) = \langle \eta(x_0, t_0)\eta(x_0, t_0 + T) \rangle$$



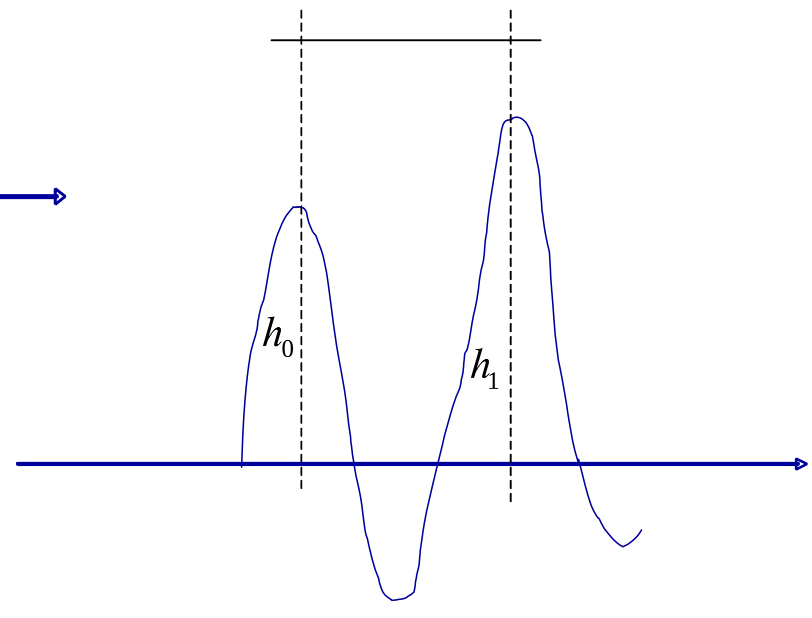
* Fedele F., **Successive wave crests in a Gaussian sea**, *Probabilistic Eng. Mechanics* 2005 (to appear)







$$T_2^* + O\left(h_0^{-1}, h_1^{-1}\right)$$

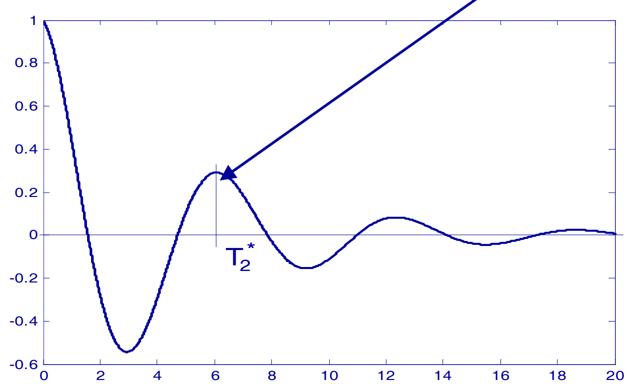


$$as \quad h_0 \to \infty, \quad h_1 \to \infty$$

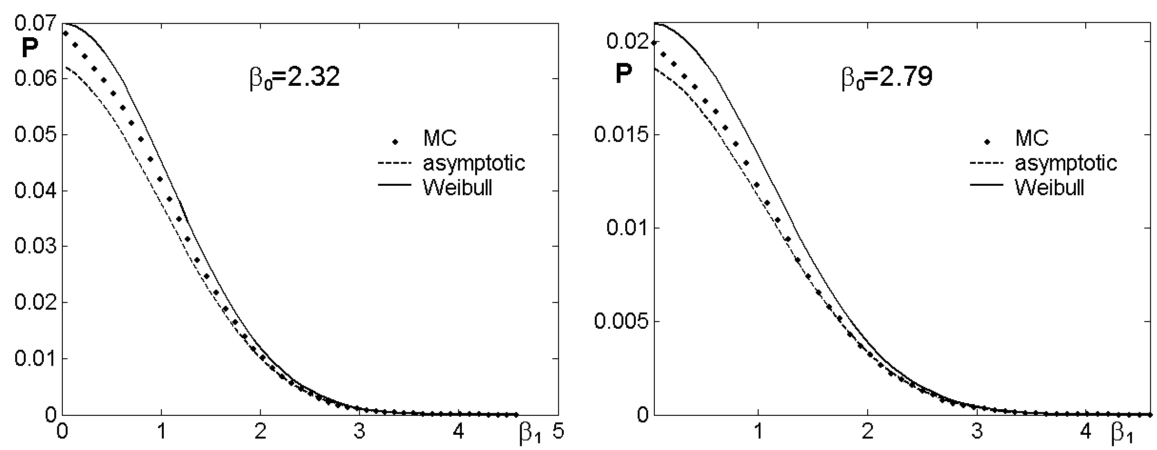
Corollary: joint probability successive wave crests*

$$p(\beta_0,\beta_1)=\frac{\beta_0\beta_1}{1-k^2}\exp\left[-\frac{\beta_0^2+\beta_1^2}{2\left(1-k^2\right)}\right]I_0\left(\frac{k\beta_0\beta_1}{1-k^2}\right)$$

Bivariate Weibull



Monte Carlo Simulations



* Fedele F., **Successive wave crests in a Gaussian sea**, *Probabilistic Eng. Mechanics* 2005 (to appear)

Space-time domain analysis

What happens in the neighborhood of a point $\mathbf{x}_0$
if two large successive wave crests are recorded in time at $\mathbf{x}_0$?

$$\Pr\left[\begin{array}{l}\eta(\mathrm{x}_0+\mathrm{X},t_0+T)\in(u,u+du) \\ \\ \textit{conditioned to} \quad \eta(\mathrm{x}_0,t_0)=h_1,\ \eta(\mathrm{x}_0,t_0+T_2^*)=h_2\end{array}\right]$$

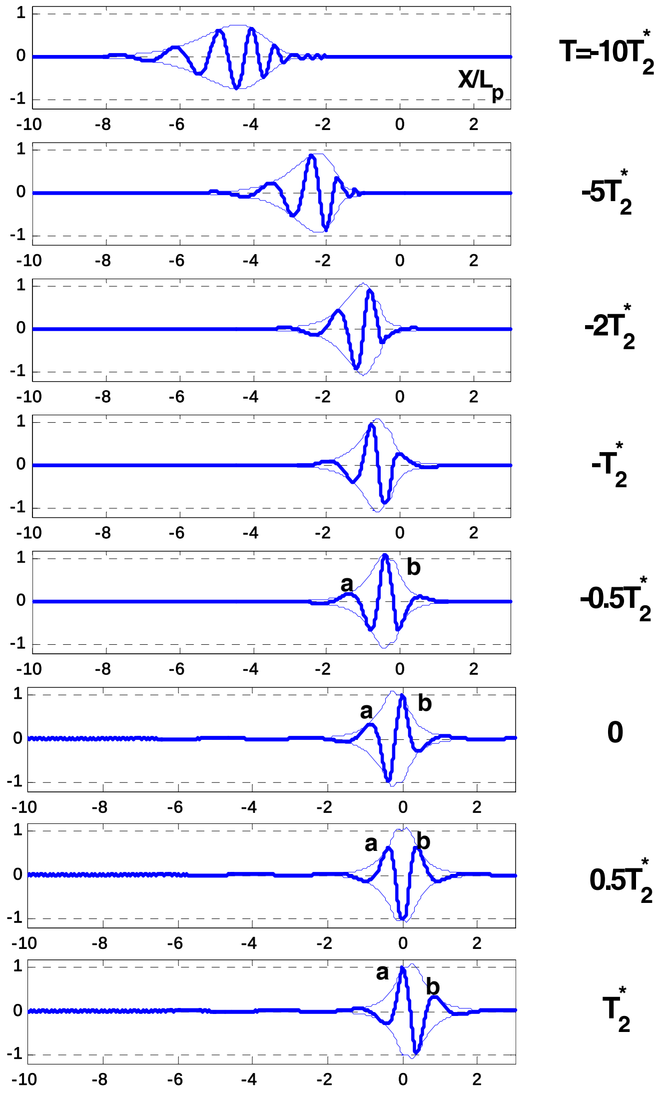
$$\Downarrow \qquad \frac{h_1}{\sigma}\to\infty, \qquad \frac{h_2}{\sigma}\to\infty$$

$$\delta\left(u-\eta_c\left(\mathbf{x}_0+\mathbf{X},t_0+T\right)\right)$$

$$\eta_c(\mathbf{X},T)=\frac{\psi(0)h_1-h_2\psi(T_2^*)}{\psi^2(0)-\psi(T_2^*)^2}\,\Psi(\mathbf{X},T)+\frac{\psi(0)h_2-h_1\psi(T_2^*)}{\psi^2(0)-\psi(T_2^*)^2}\,\Psi(\mathbf{X},T-T_2^*)$$

SPACE-TIME covariance $\quad \Psi(\mathbf{X},T)=\langle\eta(x_0,t_0)\eta(x_0+\mathbf{X},t_0+T)\rangle$

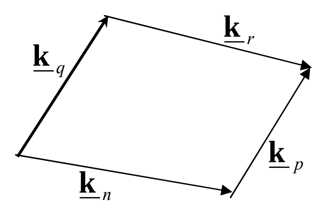
Linear wave group dynamics



Nonlinear evolution of the linear wave group

$$\eta(\underline{\mathbf{x}},t)=\frac{1}{\pi}\sum_n\sqrt{\frac{\omega_n}{2g}}\,B_n(t)\exp\left(i\,\underline{\mathbf{k}}_n\cdot\underline{\mathbf{x}}\right)+c.c.$$

$$\frac{dB_n}{dt}+i\omega_nB_n=-i\sum_{p,q,r}T_{npqr}\delta_{n+p-q-r}B_p^*B_qB_r$$



Quartet interaction

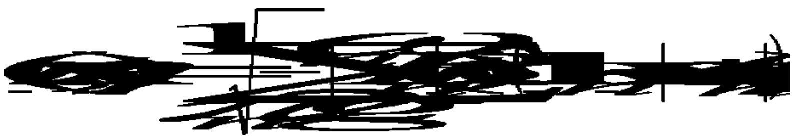
$$\underline{\mathbf{k}}_n+\underline{\mathbf{k}}_p=\underline{\mathbf{k}}_q+\underline{\mathbf{k}}_r$$

Conserved quantities : Hamiltonian , wave action and momentum

$$\mathbf{H}=\sum_n\omega_nB_n(t)B_n^*(t)+\frac{1}{2}\sum_{n,p,q,r}T_{npqr}\delta_{n+p-q-r}B_n^*(t)B_p^*(t)B_q(t)B_r(t)$$

$$\mathbf{M}=\sum_n\mathbf{k}_nB_n(t)B_n^*(t)\qquad\qquad \mathbf{A}=\sum_nB_n(t)B_n^*(t)$$

Sufficient conditions to have an extreme crest*



At (x=0, t=0) we impose that all the harmonic components are in phase (focusing)

$$\varphi_n(0) = 0 \qquad n = 1,....N$$

Constrained optimization problem

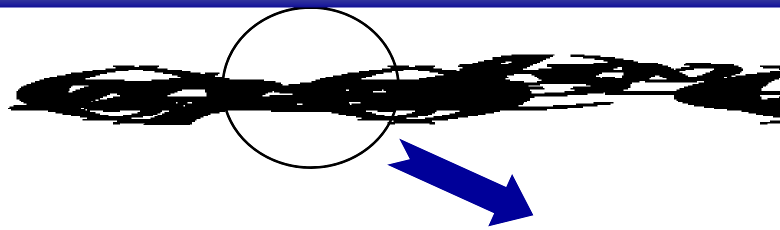
$$H_{NL} = \max \frac{1}{\pi}\sum_n \sqrt{\frac{\omega_n}{2g}}\left|B_n(0)\right|$$

Hamiltonian , wave action and momentum are conserved

$$H_{NL} = (1+\lambda)H_L \qquad \lambda = \lambda\left(\left\{\left|B_n(0)\right|\right\}\right)$$

*Fedele F. **The occurrence of Extreme crests and the nonlinear wave-wave interaction in random seas**, PROCEEDINGS of XIV International Offshore and Polar Engineering Conference (ISOPE) Toulon, FRANCE may 23-28 2004

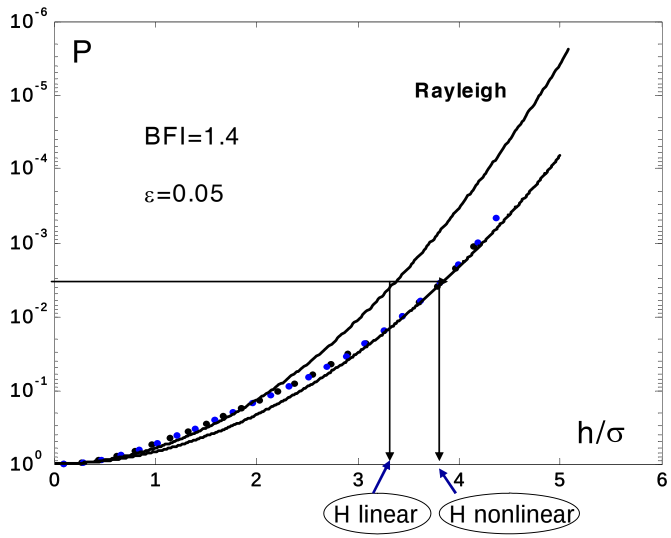
The narrow band limit*
The nonlinear Schrodinger (NLS) equation

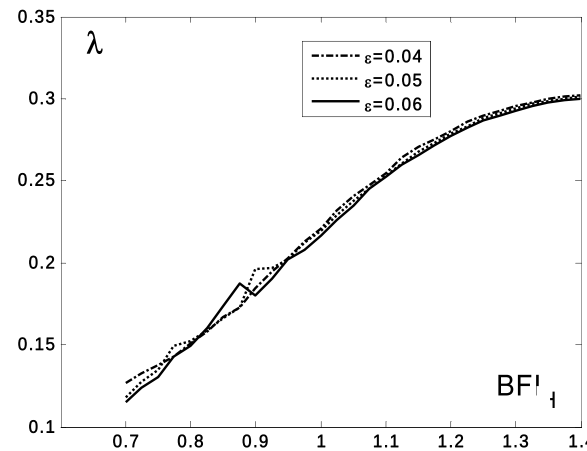


Particular case of the
ZAKHAROV EQUATION

$$\frac{\partial A}{\partial t} + i\frac{\partial^2 A}{\partial x^2} + iBFI^2 |A|^2 A = 0$$

$$\Pr\left(H_{\max} > h\right) = \exp\left[-\frac{h^2}{2(1+\lambda)^2\sigma^2}\right]$$





Intermittency (FERMI-ULAM PASTA recurrence)

*Fedele F. **The occurrence of Extreme crests and the nonlinear wave-wave interaction in random seas**, PROCEEDINGS of XIV International Offshore and Polar Engineering Conference (ISOPE) Toulon, FRANCE may 23-28 2004

Recurrent solutions* & Intermittency

$$\text{Rayleigh quotient} \quad Q(t)=\frac{\int\left|\frac{\partial A}{\partial x}\right|^2 dx}{\int |A|^2 dx}\le const \qquad \text{bounded in time}$$

[illegible]

SPECTRAL BANDWIDTH

$$Q(t)=\frac{\sum_n k_n^2|a_n(t)|^2}{\sum_n |a_n(t)|^2}\approx \Delta K^2(t) \qquad \text{bounded in time}$$

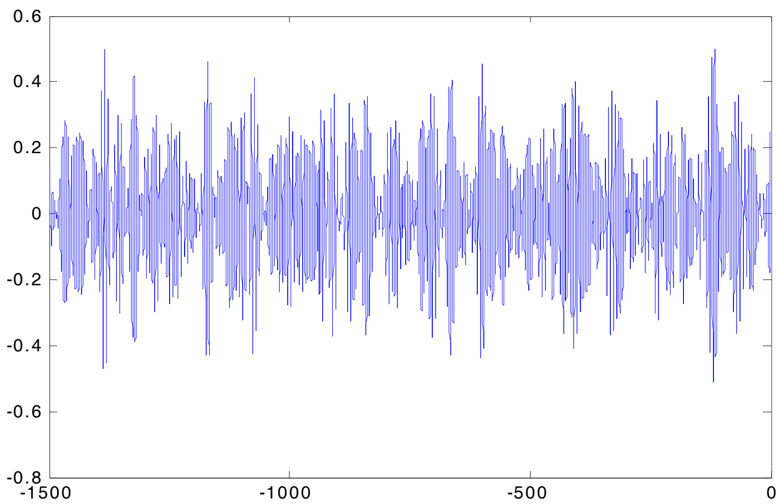
IF ENERGY CONTENT OF SMALL SCALES IS SMALL
IT REMAINS SMALL FOR ALL THE TIME

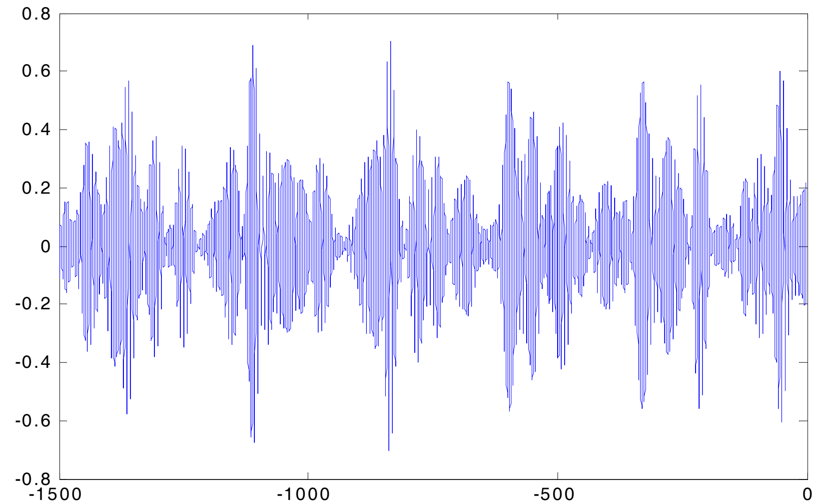
ENERGY PERPETUALLY DISTRIBUTED BETWEEN
FINITE SET OF MODES → RECURRENT SOLUTIONS

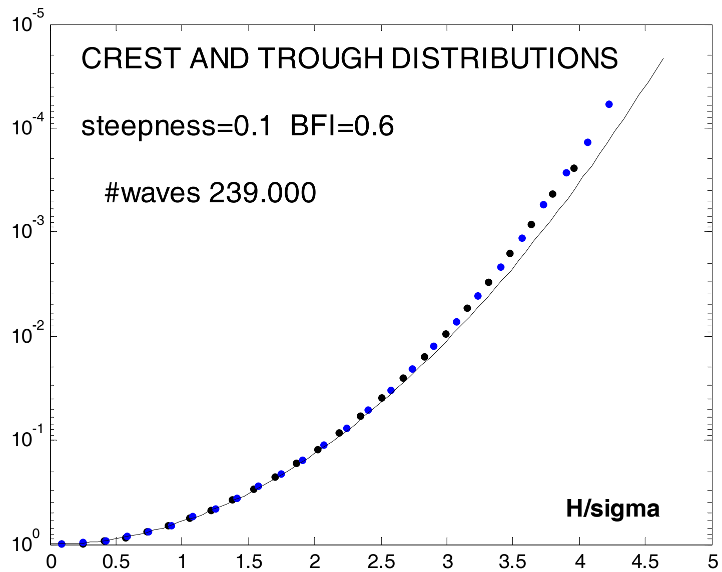
BENJAMIN-FEIR INSTABILITY & FERMI-ULAM PASTA RECURRENCE

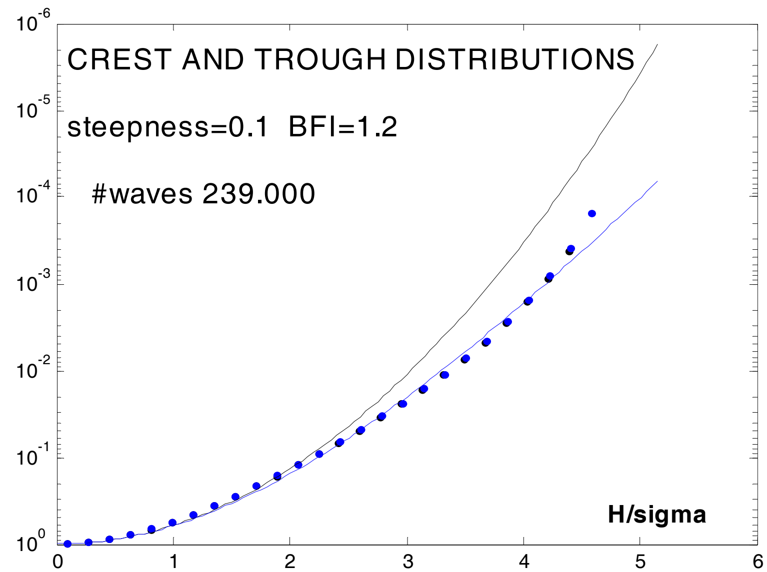
*Thyagaraja. **Recurrent motions in certain continuum dynamical systems** Physics of Fluids 22(11) 1979, pp. 2093-2096

Monte-Carlo simulations





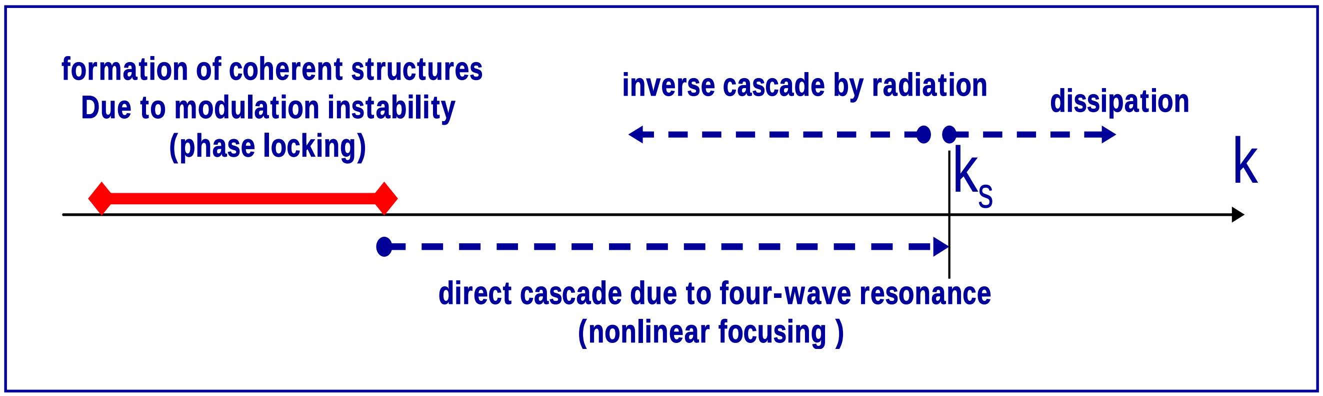


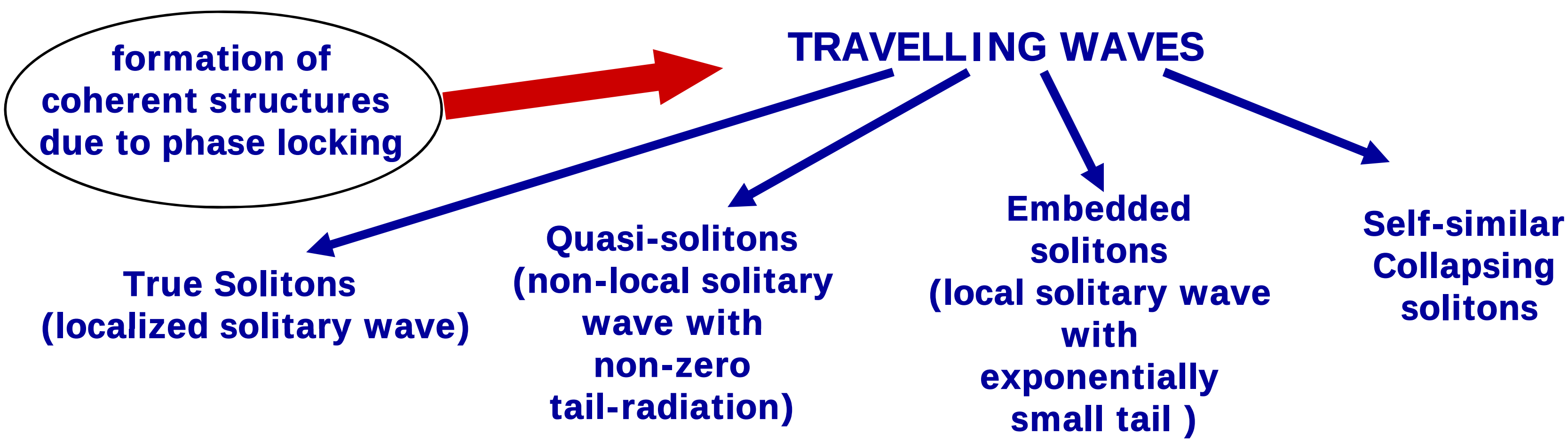


Quasi-solitonic wave turbulence*

$$i\frac{\partial \hat{a}_k}{\partial t} = \omega(k)\hat{a}_k + \iiint T_{123k}\hat{a}_1\hat{a}_2\hat{a}_3^*\delta\left(k_1 + k_2 - k_3 - k\right)dk_1 dk_2 dk_3$$

$$A(x,t) = \int \hat{a}_k(t)\exp(i\,k\,x)dk$$



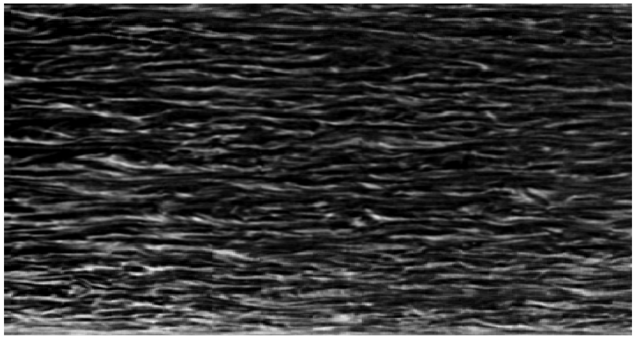


*Cai, Majda, Laughlin & Tabak, **Dispersive wave turbulence in one dimension** PHYSICA D 152-153 (2001) 551-572
Zakharov, Dias & Pushkarev, **One-dimensional wave turbulence** Physics reports 398 (2004) 1-65

Future research: are streaks similar to freak waves ?

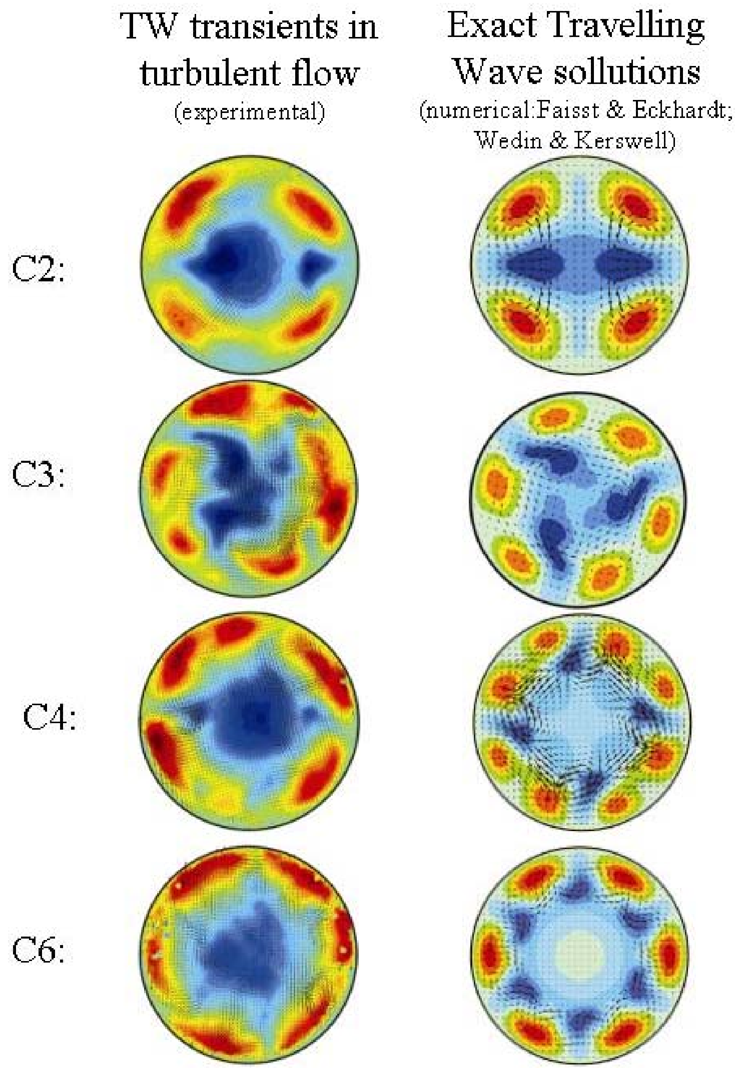
Similar underlying physical mechanism?

?

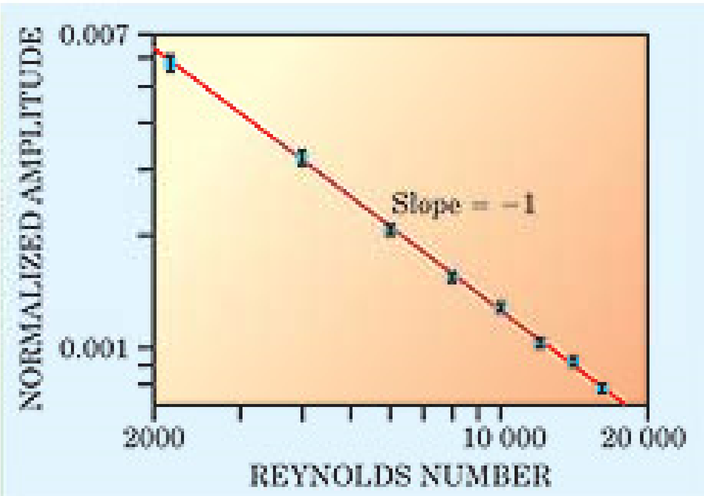


- Freak waves
  - Occur on *large* length scales and time scales

- Streaks
  - Occur on *small* length scales and time scales



From http://www-ah.wbmt.tudelft.nl/



Fabian Waleffe, Phys. Fluids, Vol. 9, pp. 883-900 (April 1997)
Hof, B., van Doorne, C.W.H., Westerweel, J., Nieuwstadt, F.T.M, Faisst, H, Eckhardt, B., Wedin, H., Kerswell, R.R., Waleffe, F. Science **305**, 1594 (2004)
Faisst, H. & Eckhardt, Phys. Rev. Lett. **91** 224502 (2003)
Wedin, H. & Kerswell, R.R. J. Fluid Mech. **508** 333 (2004)

Coupled NLS equations- a proposal

$$u(\underline{\mathbf{x}},t)=\varepsilon\sum_n a_n(t)\exp\left[i\underline{\mathbf{k}}_n\cdot\underline{\mathbf{x}}\right]+O(\varepsilon)$$

↓

$$\frac{da_n}{dt}+i\omega_n a_n=\varepsilon\sum_{p,q,r}Q_{npq}a_p^*a_q+\varepsilon^2\sum_{p,q,r}T_{npqr}a_p^*a_q a_r$$

↓

A streamwise vorticity

B streamwise velocity fluctuations

$$\frac{\partial A}{\partial t}+i\frac{\partial^2 A}{\partial x^2}+i\beta^2|A|^2\left(A+\overline{B}\right)=0$$

$$\frac{\partial B}{\partial t}+i\frac{\partial^2 B}{\partial x^2}+i\delta^2|B|^2\left(A+\overline{B}\right)=0$$

bandwidths

$$Rayleigh\ \ quotient\quad Q(t)=\frac{\int\left(\left|\frac{\partial A}{\partial x}\right|^2+\left|\frac{\partial B}{\partial x}\right|^2\right)dx}{\int\left(|A|^2+|B|^2\right)dx}\approx\frac{(\Delta K_A)^2+(\Delta K_B)^2}{2}\quad bounded\ in\ time$$

ENERGY PERPETUALLY DISTRIBUTED BETWEEN FINITE SET OF MODES
→ RECURRENT SOLUTIONS (BENJAMIN-FEIR TYPE INSTABILITY)

COMPARE STREAK STATISTICS FROM DNS SIMULATIONS WITH THE NLS STATISTICS

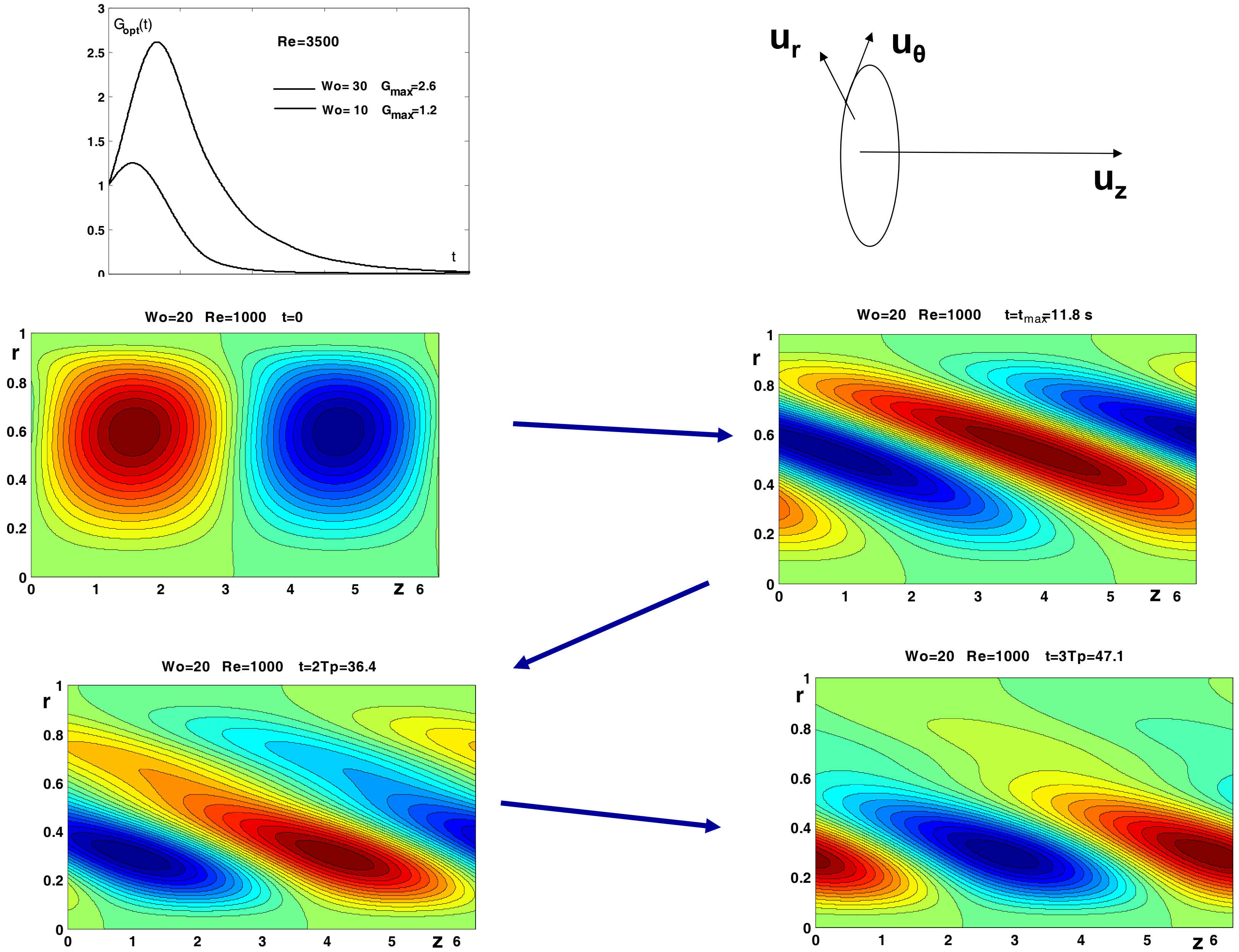
Conclusions

- ***Weakly nonlinear effects***
  - **COHERENT STRUCTURES FORMED DUE TO MODULATION INSTABILITY OF A LINEAR WAVE GROUP (PHASE LOCKING)**
  - **RECURRENT SOLUTIONS CAN OCCUR (slowly varying envelope)**

- ***Strong nonlinear effects***
  - **QUASI-SOLITONIC TURBULENCE (Freak waves occur)**

- ***Future research***
  - **SIMILARITIES BETWEEN STREAKS AND FREAK WAVES**
  - **POSSIBLE ENVELOPE EQUATIONS TO DESCRIBE INITIAL STAGE OF WEAKLY NONLINEAR STREAK DYNAMICS DUE TO MODULATION INSTABILITY**

Optimal Perturbations: Maximum Energy Growth

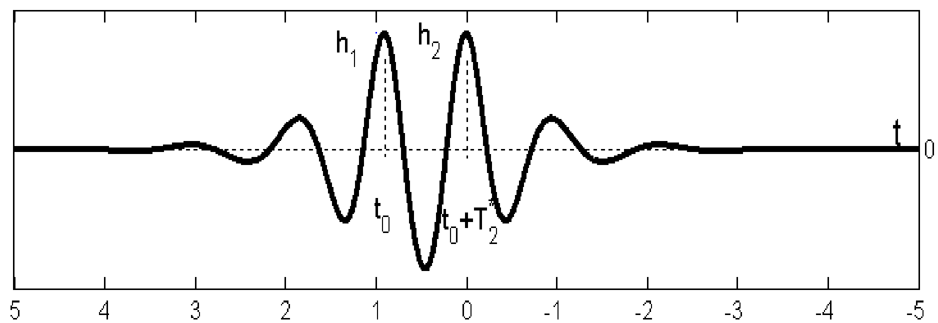


TIME DOMAIN : *THE CONDITIONS ARE SUFFICIENT*

$$\frac{h_1}{\sigma} \to \infty, \quad \frac{h_2}{\sigma} \to \infty$$

$$\Pr\begin{bmatrix} \eta(t_0+T) \in (u, u+du) \\ conditioned\ to\ \ \eta(t_0)=h_1,\ \eta(t_0+T_2^*)=h_2 \end{bmatrix} \quad \to \quad \delta[u-\eta_c(t_0+T)]$$

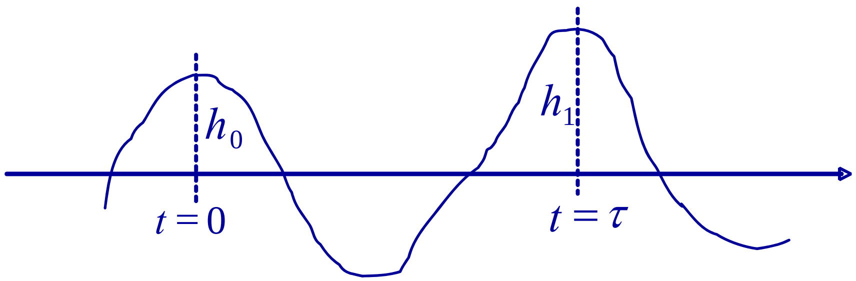
$$\eta_c(t_0+T) = \frac{h_1 - h_2\,\psi(T_2^*)/\psi(0)}{1-\left(\psi(T_2^*)/\psi(0)\right)^2}\ \psi(T) + \frac{h_2 - h_1\,\psi(T_2^*)/\psi(0)}{1-\left(\psi(T_2^*)/\psi(0)\right)^2}\ \psi(T-T_2^*)$$



TIME DOMAIN : *THE CONDITIONS ARE NECESSARY*

$$EX_c(h_1, h_2, \tau)$$

Expected number of local maxima of the surface displacement $\eta(t)$ **of amplitude** h_0 **which are followed by a local maximum with amplitude** h_1 **after a time lag** τ



$$\beta_0 = \frac{h_0}{\sigma} \to \infty, \qquad \beta_1 = \frac{h_1}{\sigma} \to \infty$$

$$EX_{s.c.}(h_1, h_2, \tau) = \begin{cases} EX_c(h_1, h_2, T_2^*)\exp\left[-\frac{1}{2}K^*\delta\tau^2\right] & (\delta\tau) \propto O\left(\beta_0^{-1}, \beta_1^{-1}\right) \\ 0 \quad elsewhere & \end{cases}$$

Corollary : joint probability of successive wave crests

$$p(\beta_0,\beta_1) \propto \int_0^\infty EX_c(\beta_0,\beta_1,\tau)\,d\tau$$

$$p(\beta_0,\beta_1) = \frac{1+\ddot{\psi}_2^*\psi_2^*}{\sqrt{-2\pi\ddot{\psi}_2^*\left(1-\psi_2^{*2}\right)^3}}\exp\left[-\frac{\beta_0^2+\beta_1^2-2\psi_2^*\beta_0\beta_1}{2\left(1-\psi_2^{*2}\right)}\right]\sqrt{\left(-\beta_0+s\beta_1\right)\left(-\beta_1+s\beta_2\right)}$$

$$p(\beta_0,\beta_1) = \frac{\beta_0\beta_1}{1-k^2}\exp\left[-\frac{\beta_0^2+\beta_1^2}{2\left(1-k^2\right)}\right]I_0\left(\frac{k\beta_0\beta_1}{1-k^2}\right)$$

Bivariate Weibull