WAVE GROUPS AND EXTREME EVENTS IN RANDOM SEAS

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Freak waves, rogue waves and giant waves

Nonlinear water waves

Gaussian seas
and extreme waves

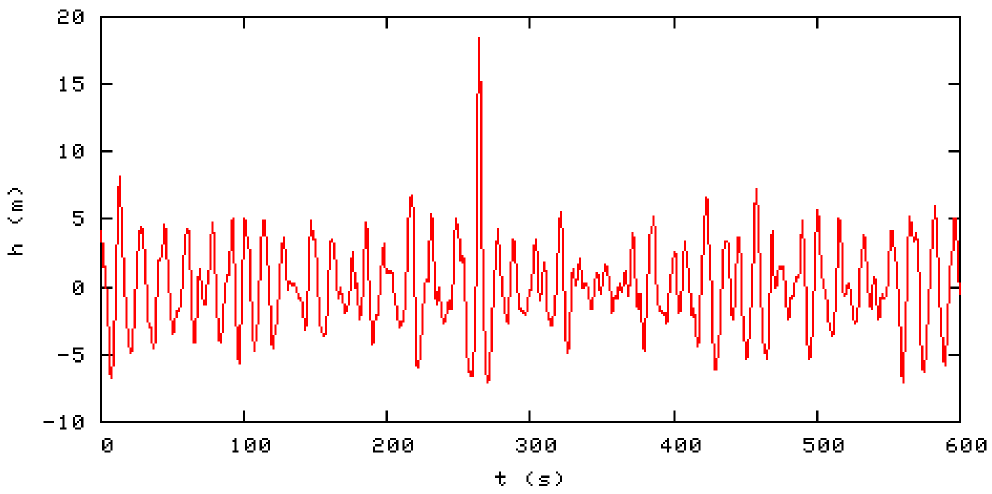
Freak waves

Rogue waves

Giant waves

Extreme waves

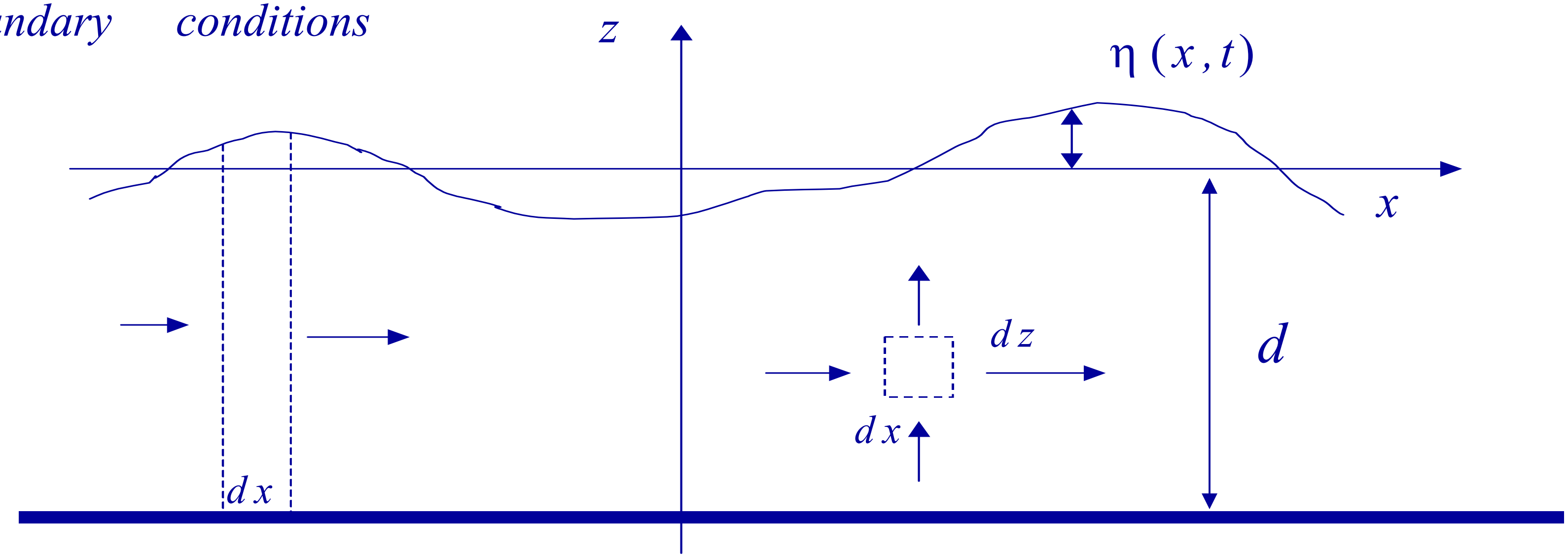
DRAUPNER EVENT JANUARY 1995



STOKES EQUATIONS FOR REGULAR WAVES

$$
\begin{cases}
\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial z^2} = 0 \\[2ex]
\left(\dfrac{\partial \Phi}{\partial z}\right)_{z=\eta} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \eta}{\partial x}\left(\dfrac{\partial \Phi}{\partial x}\right)_{z=\eta} \\[2ex]
\left(\dfrac{\partial \Phi}{\partial t}\right)_{z=\eta} + \dfrac{1}{2}\left[\left(\dfrac{\partial \Phi}{\partial x}\right)^2 + \left(\dfrac{\partial \Phi}{\partial z}\right)^2\right]_{z=\eta} + g\eta = f(t) \\[2ex]
\text{boundary} \quad \text{conditions}
\end{cases}
$$

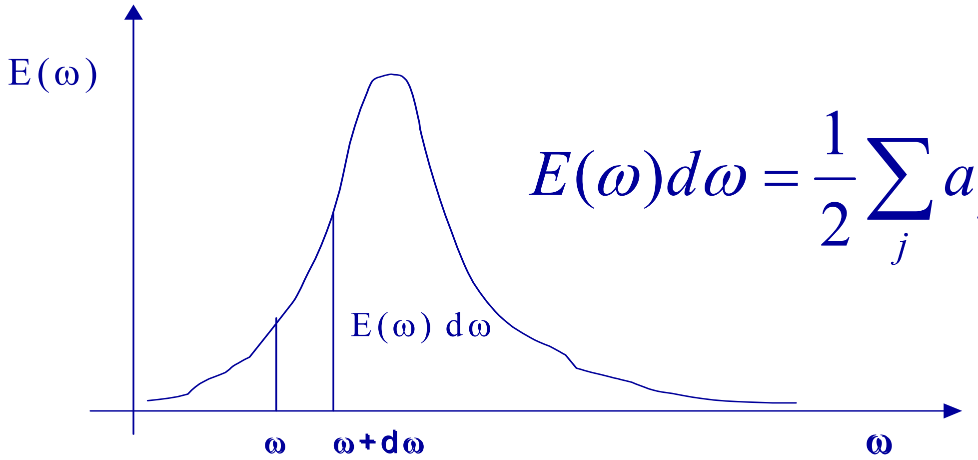
$$v_z = \frac{\partial \Phi}{\partial z} \qquad v_x = \frac{\partial \Phi}{\partial x}$$



GAUSSIAN SEAS

$$\eta(x,t) = \sum_{j=1}^{N} a_j \cos\left(k_j x + \omega_j t + \varepsilon_j\right)$$

ε_j *uniformly random in* $[0, 2\pi]$



$$E(\omega)d\omega = \frac{1}{2}\sum_j a_j^2 \qquad \omega < \omega_j < \omega + d\omega$$

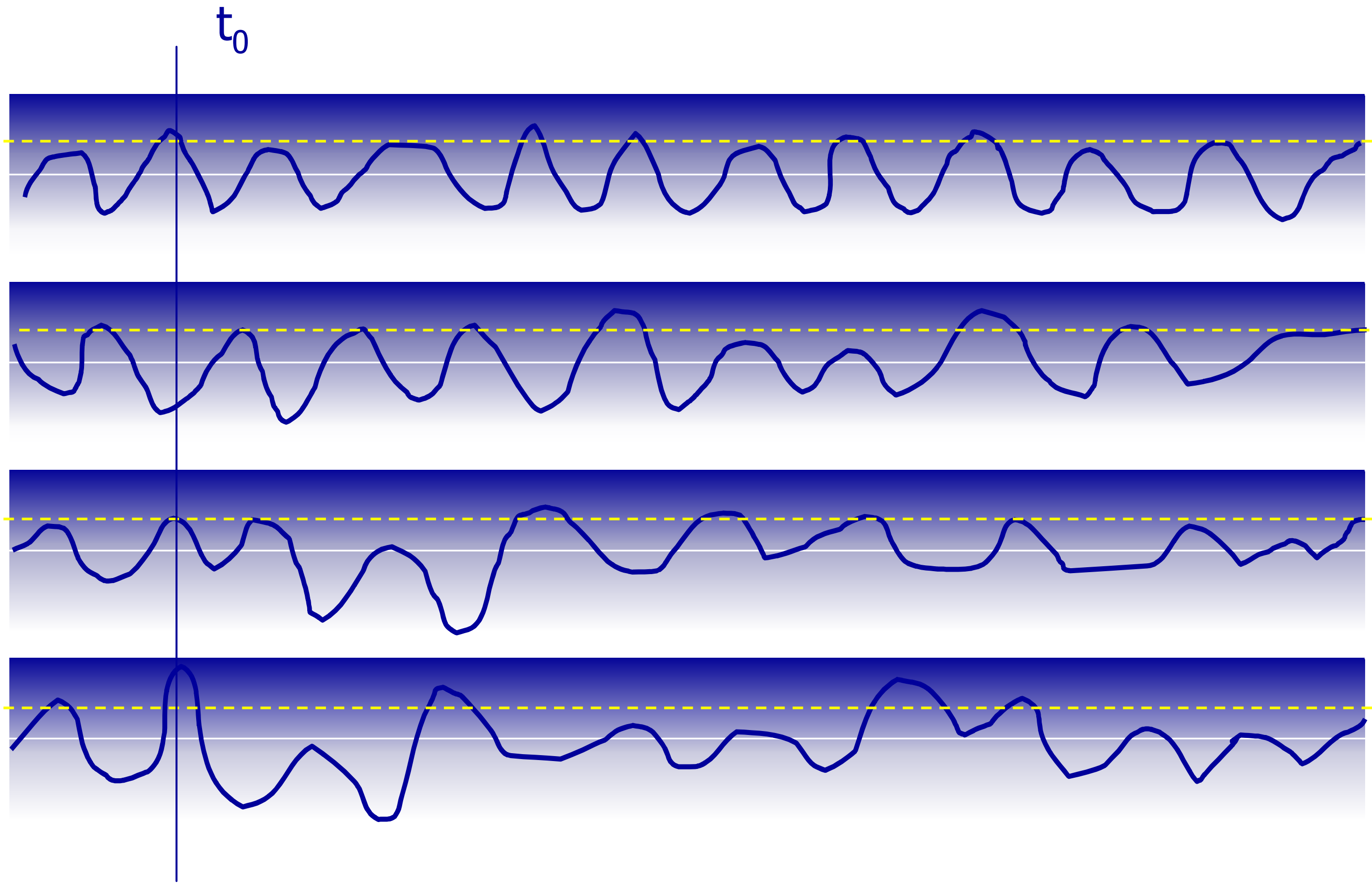
Stationarity

Ergodicity

Gaussianity

$$\Pr[\eta(t_0) > z] = \frac{\#\ \textit{realizations in which } \eta \textit{ is greater than } z \textit{ at the time } t_0}{\#\ \textit{realizations}}$$

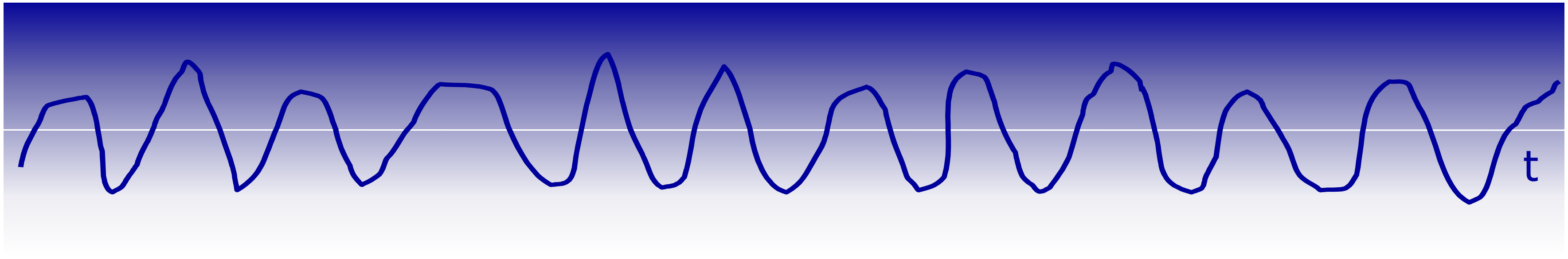
$$\Pr[\eta(t) > z] = \frac{\textit{time during which the wave elevation } \eta \textit{ is greater than } z}{\textit{total time of one realization}}$$



ERGODIC THEOREM

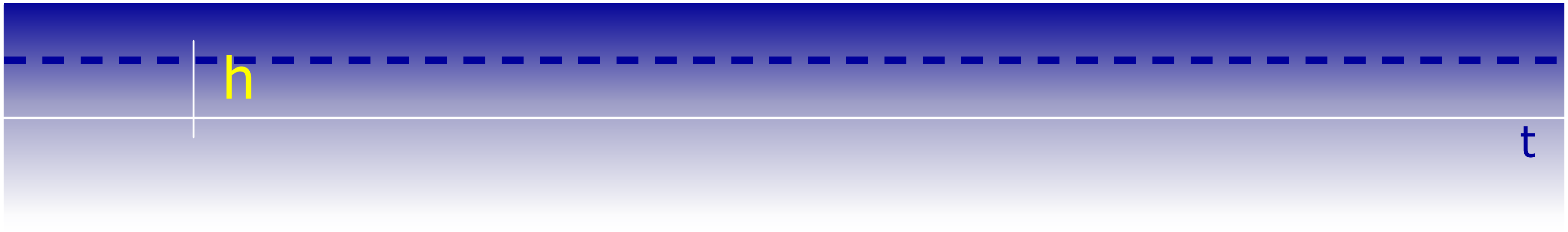
$$\eta(t) = \sum_{j=1}^{N} a_j \cos(\omega_j t + \varepsilon_j)$$

$$\overline{\eta} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(\tau) d\tau$$



$$\eta(t) = h$$

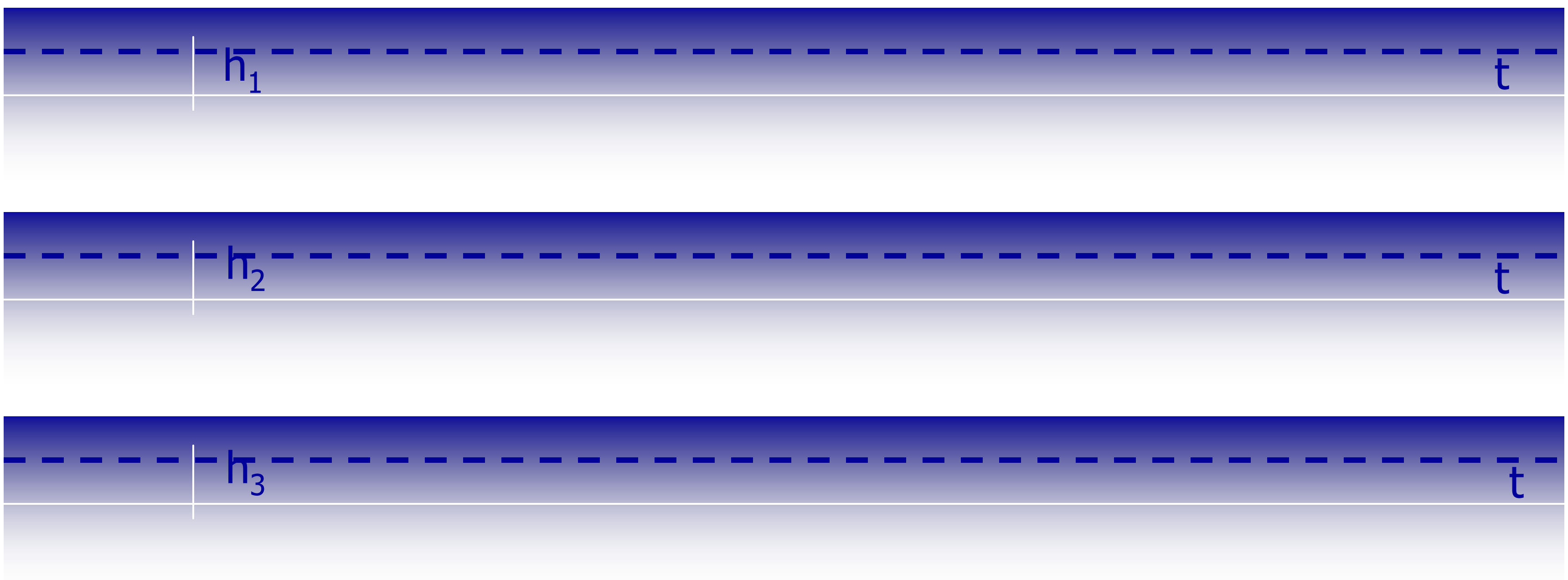
$$\overline{\eta} \neq \lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(\tau) d\tau = h$$



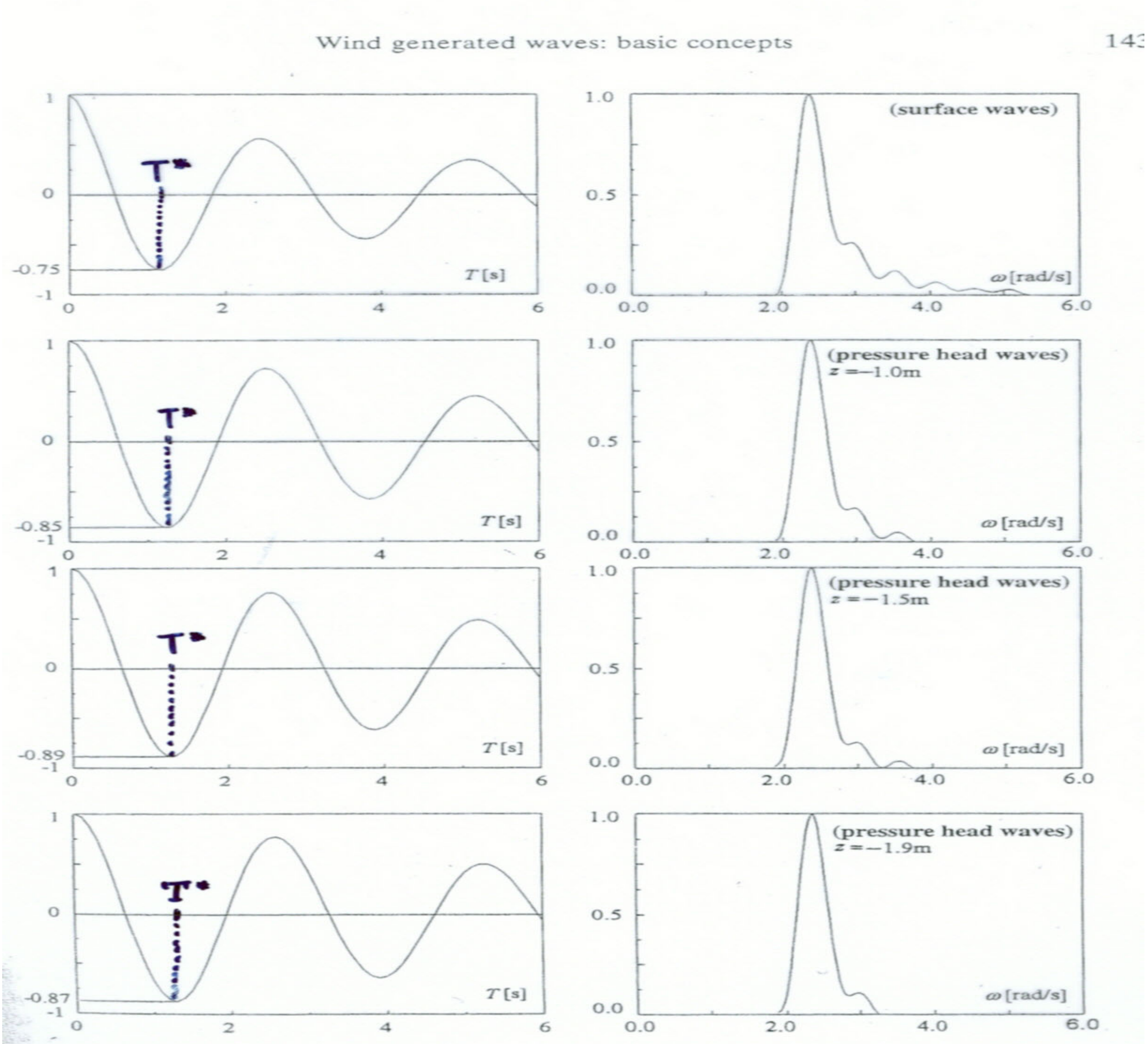
A STATIONARY GAUSSIAN NON ERGODIC PROCESS

$$\eta(t) = h$$

h constant gaussian

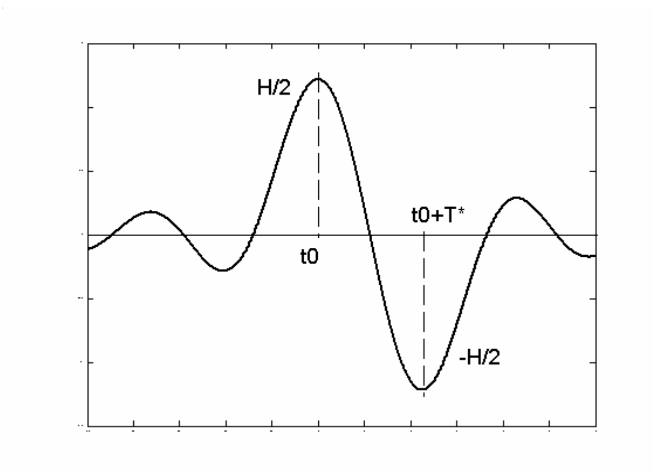


TYPICAL WAVE SPECTRA FROM MEDITERRANEAN SEA

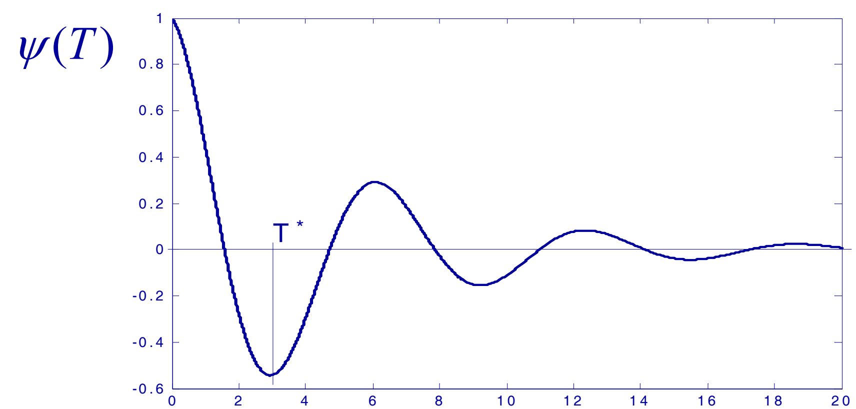


TIME DOMAIN ANALYSIS :

NECESSARY AND SUFFICIENT CONDITIONS TO HAVE A HIGH WAVE



$$\eta(0) = \frac{H}{2} \qquad \eta\left(T^*\right) = -\frac{H}{2}$$

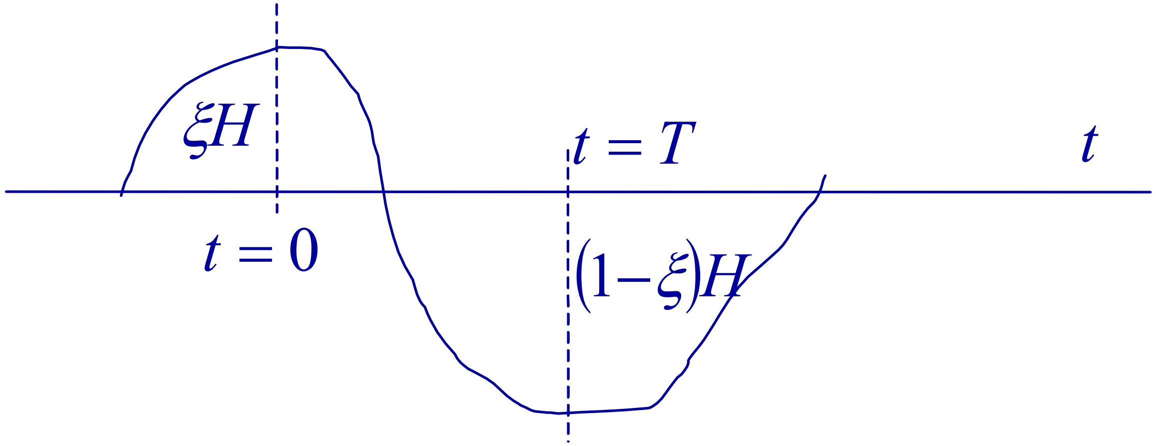


$$\psi(T) = \langle \eta(x_0, t_0)\eta(x_0, t_0 + T) \rangle$$

TIME DOMAIN : *THE CONDITIONS ARE NECESSARY*

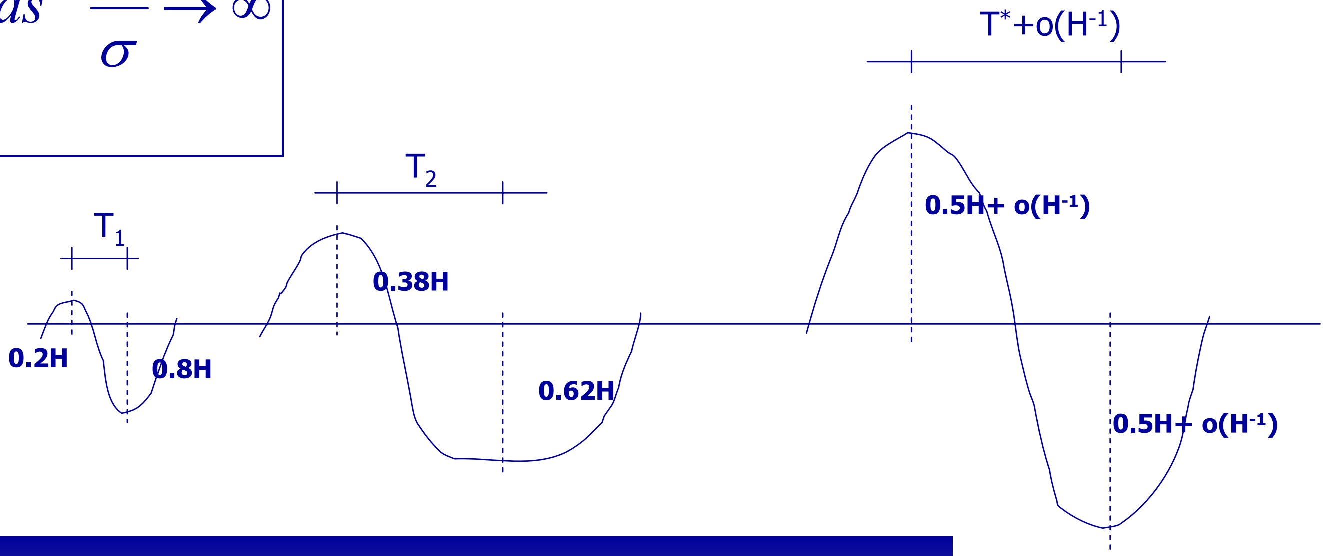
$$\eta(0) = \frac{H}{2} \qquad \eta(T^*) = -\frac{H}{2} \qquad \frac{H}{\sigma} \rightarrow \infty$$

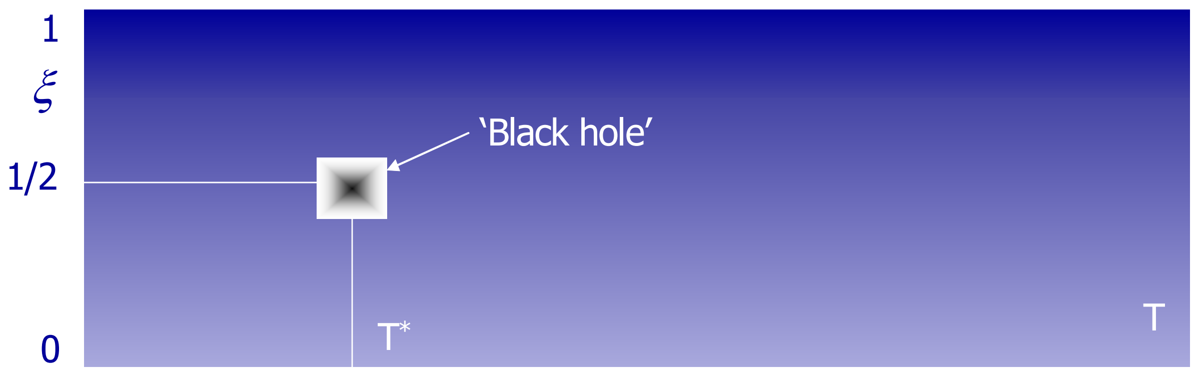
$$\Pr\left[\eta(0) = \xi H + d\eta_1 \,,\, \eta(T) = (1-\xi)H + d\eta_2\right] \qquad ?$$

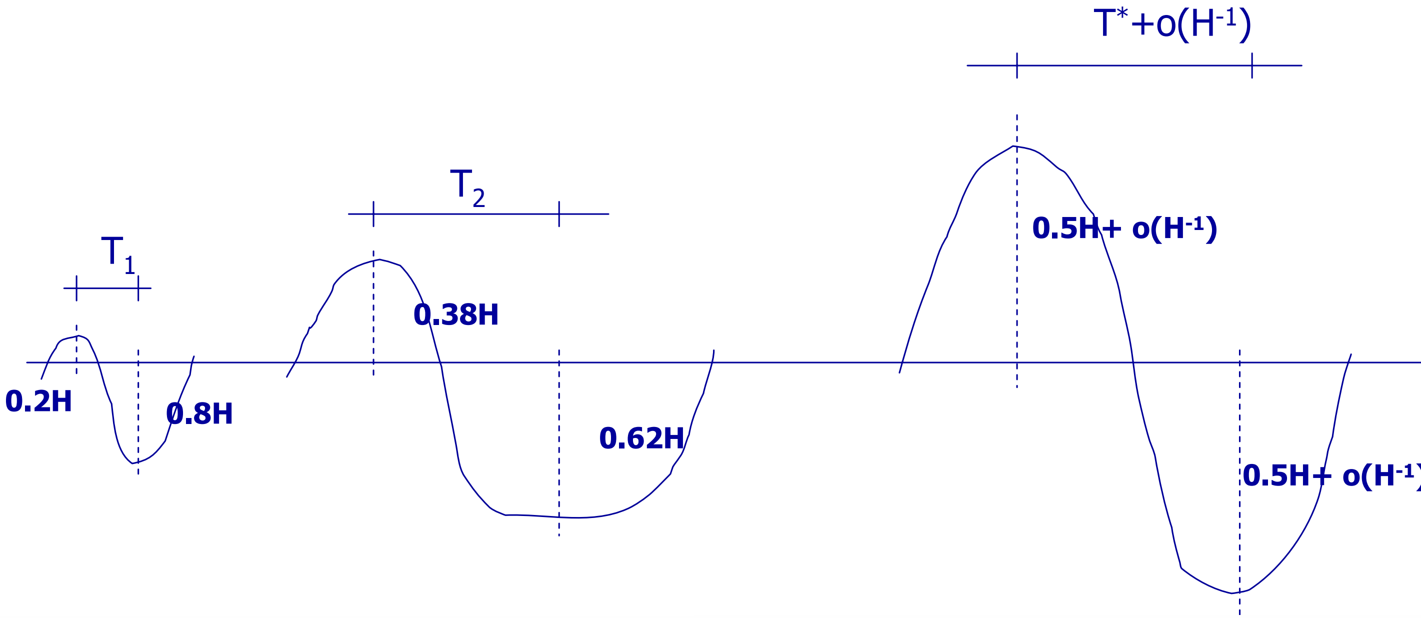


$$P(H,\xi,T) \propto \exp\left[-\frac{1}{2}\left(\frac{\sigma^2}{\sigma^2-\psi(T)}+\beta\left(\xi-\frac{1}{2}\right)^2\right)\left(\frac{H}{\sigma}\right)^2\right]$$

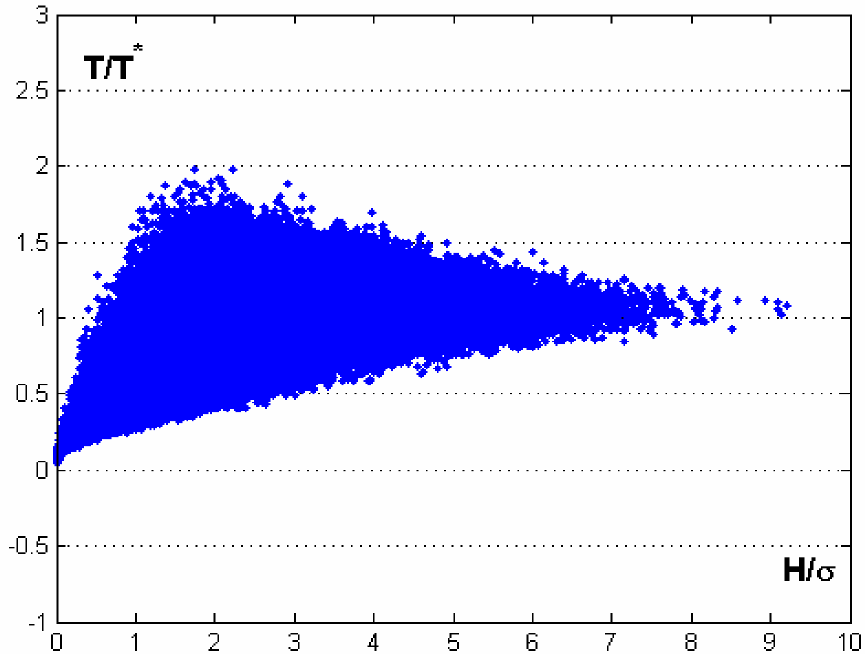
$$\frac{P(H,\xi,T)}{P\left(H,\frac{1}{2},T^*\right)} \to 0 \quad as \quad \frac{H}{\sigma} \to \infty$$



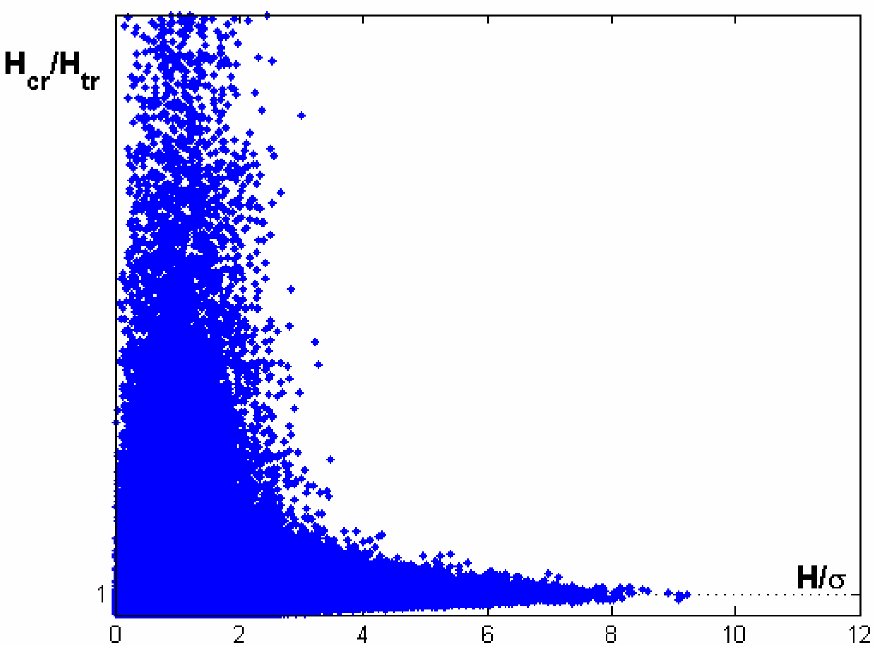




T*+o(H-1)
T2
T1
0.5H+ o(H-1)
0.38H
0.2H
0.8H
0.62H
0.5H+ o(H-1)



T/T*
H/σ

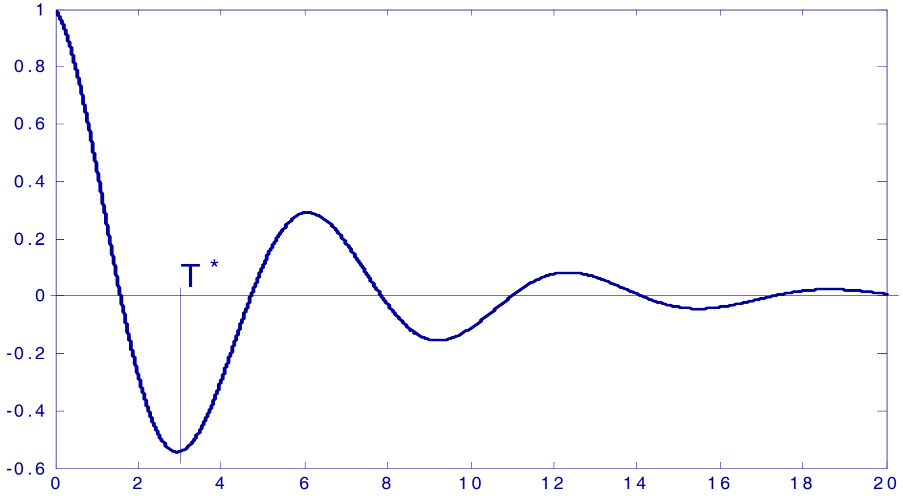


Hcr/Htr
H/σ

PROBABILITY OF EXCEEDANCE OF THE WAVE HEIGHT

Asymptotic expressions of Boccotti valid for any shape of sprectrum

$$P[H] = \exp\left[-\frac{1}{4(1+\psi^*)}\left(\frac{H}{\sigma}\right)^2\right] \quad per \quad \frac{H}{\sigma} \to \infty$$



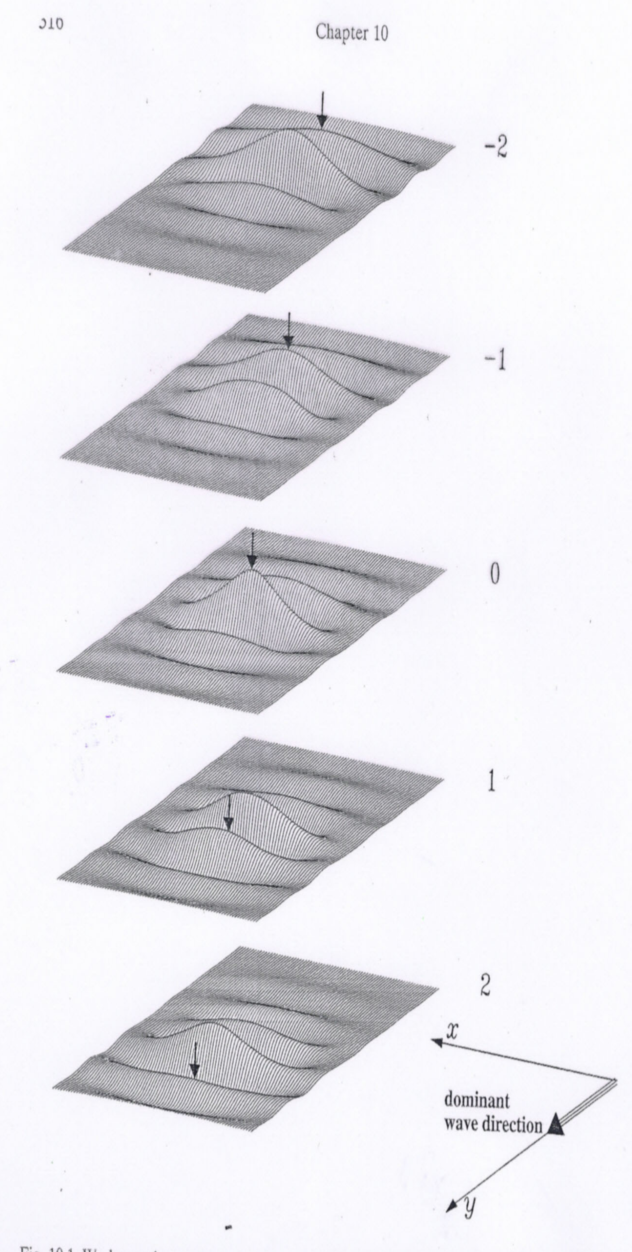
SPACE-TIME DOMAIN ANALYSIS

What happens in the neighborhood of a point $\mathbf{x}_0$ if a large crest followed by large trough are recorded in time at $\mathbf{x}_0$?

$$\Pr\left[\begin{array}{l} \eta(\mathrm{x}_0+\mathrm{X},t_0+T)\in\left(u,u+du\right) \\ \\ conditioned \quad to \\ \eta(\mathrm{x}_0,t_0)=H/2,\;\; \eta(\mathrm{x}_0,t_0+T_2^*)=-H/2 \end{array}\right]$$

$$\Big\Downarrow \qquad \frac{H}{\sigma}\rightarrow\infty$$

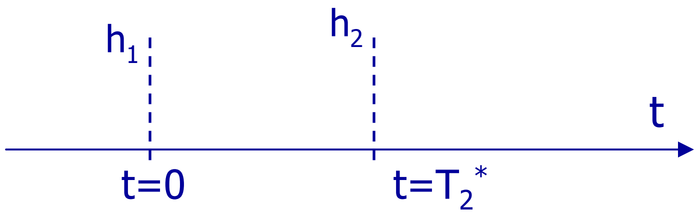
$$\delta\left(u-\eta_c\left(\mathbf{x}_0+\mathbf{X},t_0+T\right)\right)$$



TIME DOMAIN ANALYSIS : *SUCCESSIVE WAVE CRESTS*

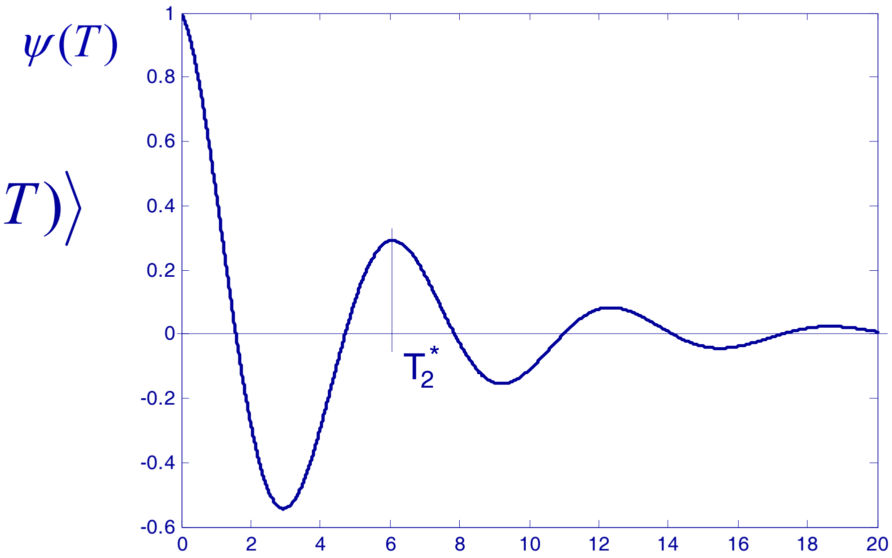
Necessary and sufficient conditions for the occurrence of two high wave crests

$$\eta(t_0) = h_1, \quad \eta(t_0 + T_2^*) = h_2$$



Autocovariance function

$$\psi(T) = \langle \eta(x_0, t_0)\eta(x_0, t_0 + T) \rangle$$

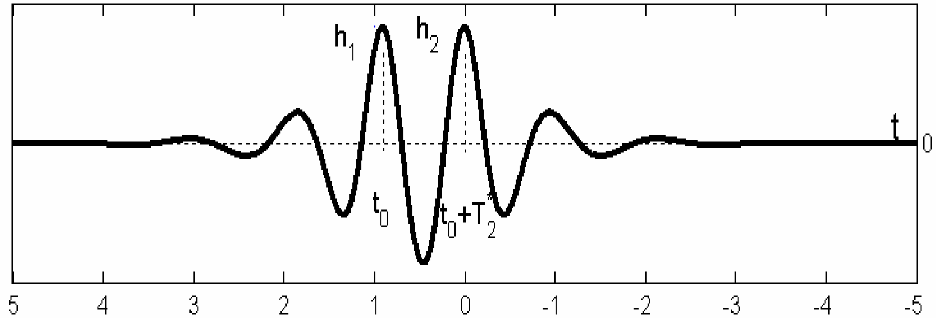


TIME DOMAIN : *THE CONDITIONS ARE SUFFICIENT*

$$\frac{h_1}{\sigma} \to \infty, \quad \frac{h_2}{\sigma} \to \infty$$

$$\Pr\begin{bmatrix} \eta(t_0+T) \in (u, u+du) \\ conditioned\ to\ \ \eta(t_0)=h_1,\ \eta(t_0+T_2^*)=h_2 \end{bmatrix} \quad \to \quad \delta[u-\eta_c(t_0+T)]$$

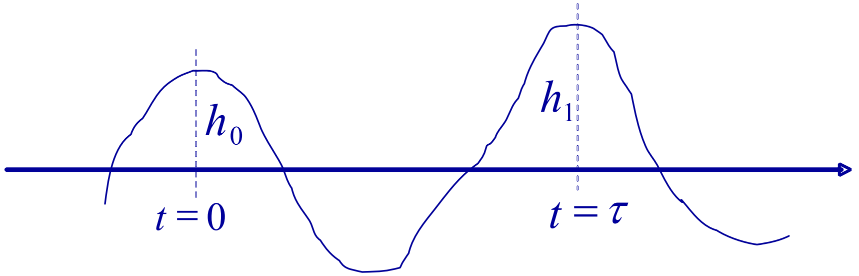
$$\eta_c(t_0+T) = \frac{h_1 - h_2\,\psi(T_2^*)/\psi(0)}{1-\left(\psi(T_2^*)/\psi(0)\right)^2}\,\psi(T) + \frac{h_2 - h_1\,\psi(T_2^*)/\psi(0)}{1-\left(\psi(T_2^*)/\psi(0)\right)^2}\,\psi(T-T_2^*)$$



TIME DOMAIN : *THE CONDITIONS ARE NECESSARY*

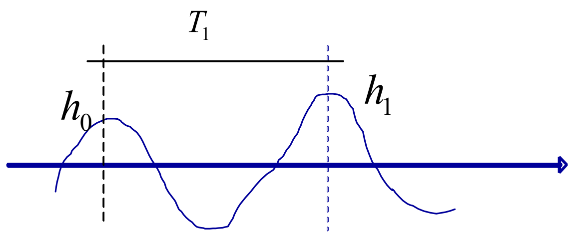
$$EX_c(h_1, h_2, \tau)$$

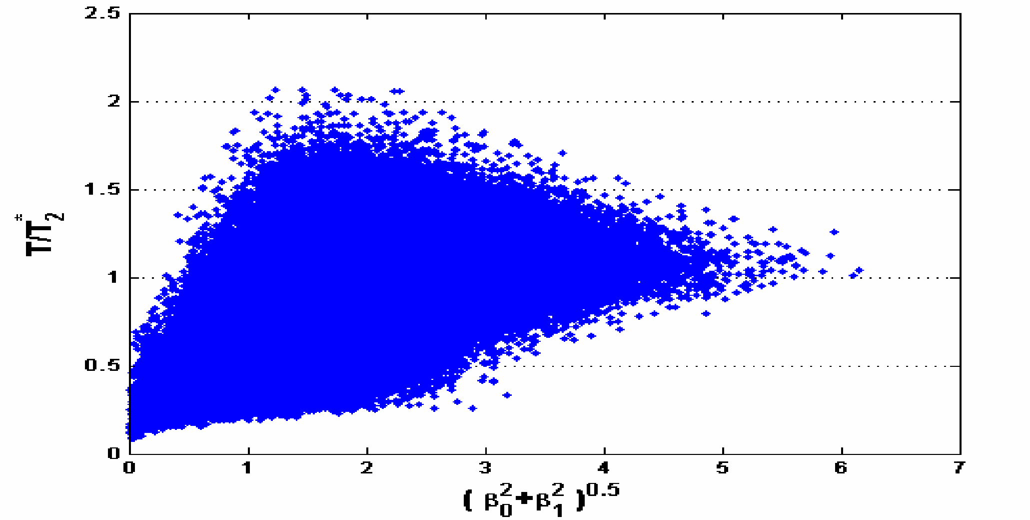
Expected number of local maxima of the surface displacement $\eta(t)$ **of amplitude** h_0 **which are followed by a local maximum with amplitude** h_1 **after a time lag** τ

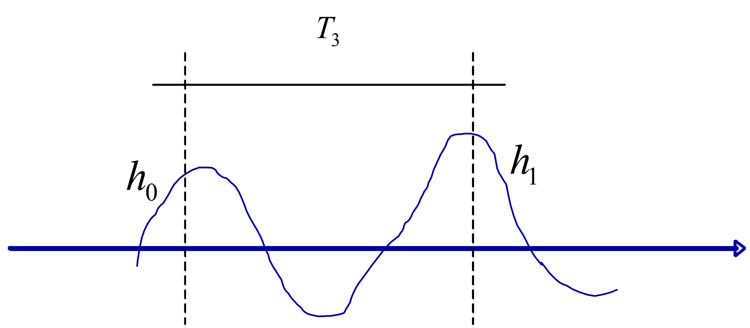


$$\beta_0 = \frac{h_0}{\sigma} \to \infty, \qquad \beta_1 = \frac{h_1}{\sigma} \to \infty$$

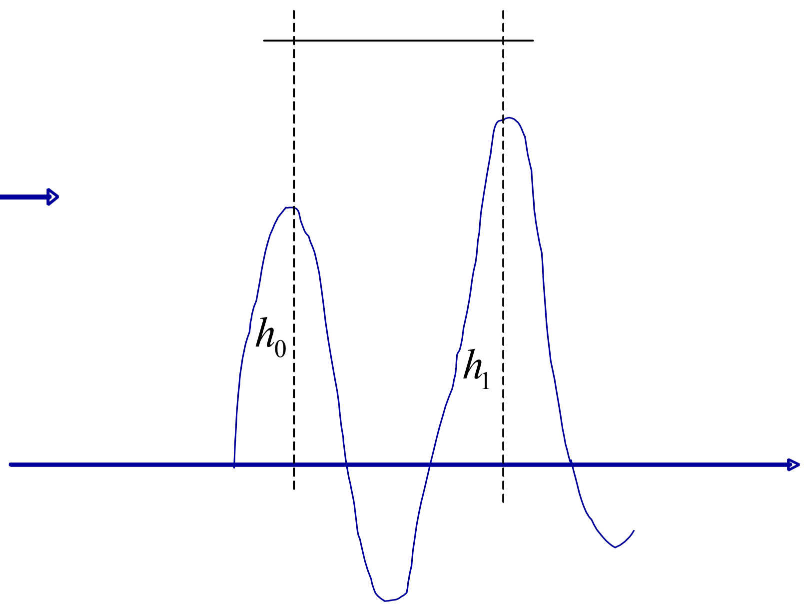
$$EX_{s.c.}(h_1, h_2, \tau) = \begin{cases} EX_c(h_1, h_2, T_2^*)\exp\left[-\frac{1}{2}K^*\delta\tau^2\right] & (\delta\tau) \propto O\left(\beta_0^{-1}, \beta_1^{-1}\right) \\ 0 & elsewhere \end{cases}$$







$$T_2^* + O\left(h_0^{-1}, h_1^{-1}\right)$$



$$as \quad h_0 \to \infty, \quad h_1 \to \infty$$

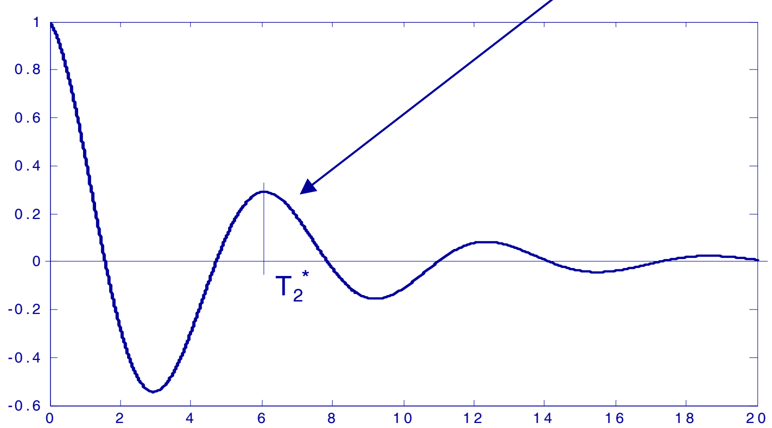
Corollary : joint probability of successive wave crests

$$p(\beta_0,\beta_1) \propto \int_0^\infty EX_c(\beta_0,\beta_1,\tau)\,d\tau$$

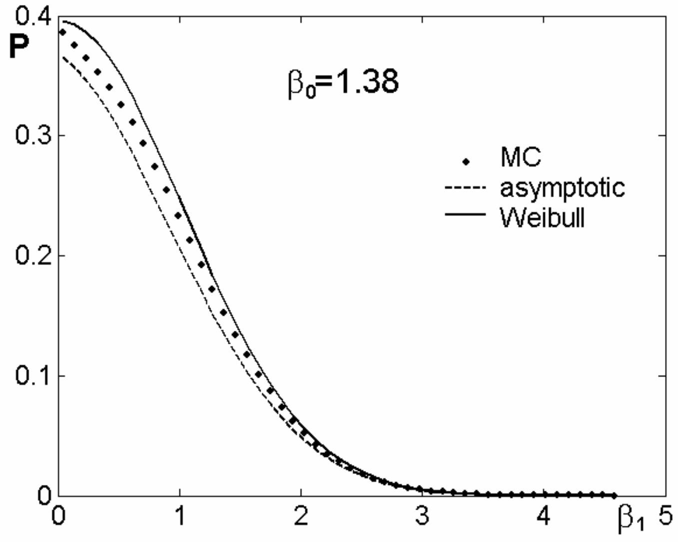
$$p(\beta_0,\beta_1) = \frac{1+\ddot{\psi}_2^*\psi_2^*}{\sqrt{-2\pi\ddot{\psi}_2^*\left(1-\psi_2^{*2}\right)^3}}\exp\left[-\frac{\beta_0^2+\beta_1^2-2\psi_2^*\beta_0\beta_1}{2\left(1-\psi_2^{*2}\right)}\right]\sqrt{\left(-\beta_0+s\beta_1\right)\left(-\beta_1+s\beta_2\right)}$$

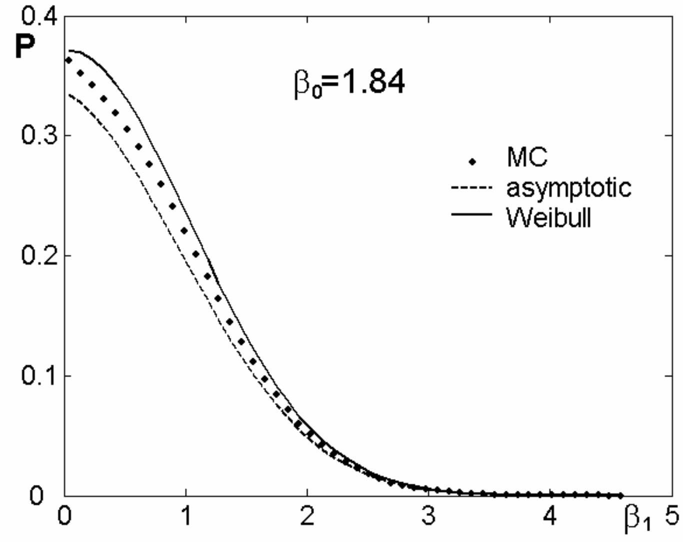
$$p(\beta_0,\beta_1) = \frac{\beta_0\beta_1}{1-k^2}\exp\left[-\frac{\beta_0^2+\beta_1^2}{2\left(1-k^2\right)}\right]I_0\left(\frac{k\beta_0\beta_1}{1-k^2}\right)$$

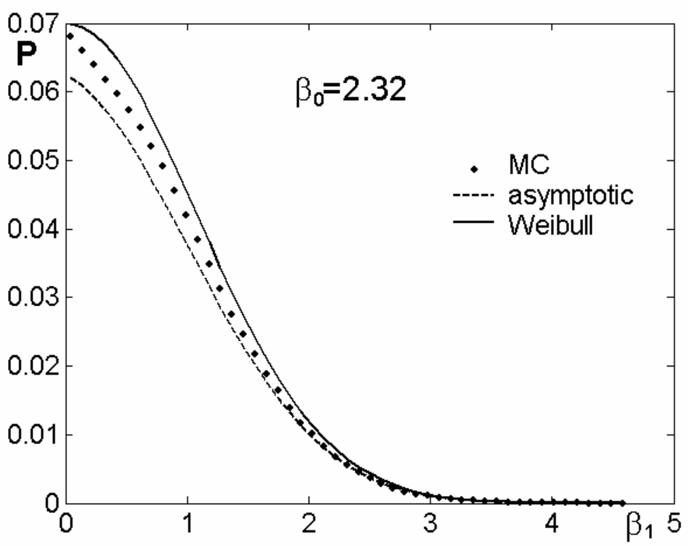
Bivariate Weibull

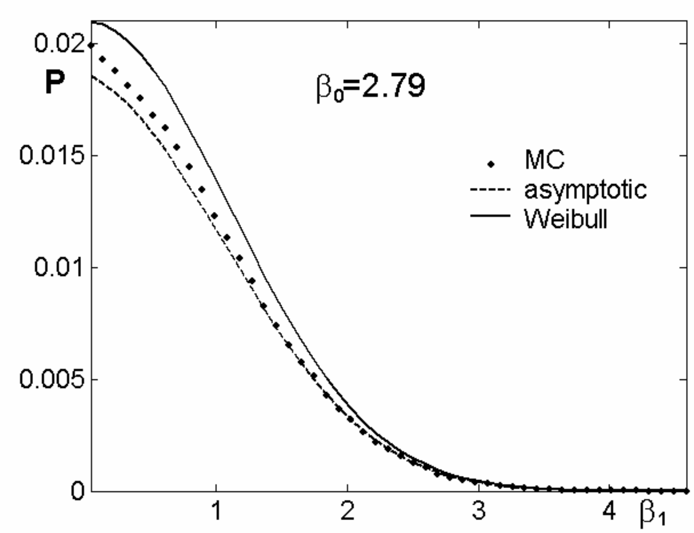


MONTE CARLO SIMULATIONS OF GAUSSIAN SEAS









SPACE-TIME DOMAIN ANALYSIS

What happens in the neighborhood of a point $\mathbf{x}_0$
if two large successive wave crests are recorded in time at $\mathbf{x}_0$?

$$\Pr\left[\begin{array}{l} \eta(\mathrm{x}_0+\mathrm{X},t_0+T)\in\left(u,u+du\right) \\ \\ conditioned\ \ to \quad \eta(\mathrm{x}_0,t_0)=h_1,\ \ \eta(\mathrm{x}_0,t_0+T_2^*)=h_2 \end{array}\right]$$

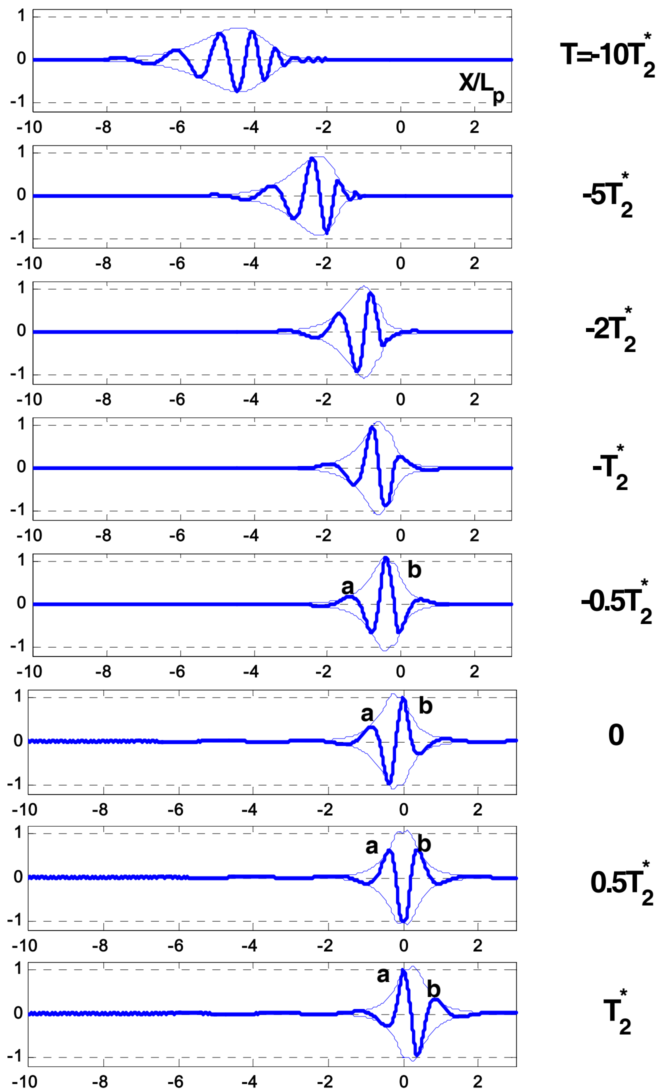
$$\Big\Downarrow \qquad \frac{h_1}{\sigma}\to\infty,\qquad \frac{h_2}{\sigma}\to\infty$$

$$\delta\left(u-\eta_c\left(\mathbf{x}_0+\mathbf{X},t_0+T\right)\right)$$

$$\eta_c(\mathbf{X},T)=\frac{\psi(0)h_1-h_2\psi(T_2^*)}{\psi^2(0)-\psi(T_2^*)^2}\ \Psi(\mathbf{X},T)+\frac{\psi(0)h_2-h_1\psi(T_2^*)}{\psi^2(0)-\psi(T_2^*)^2}\ \Psi(\mathbf{X},T-T_2^*)$$

SPACE-TIME covariance $\qquad \Psi(\mathbf{X},T)=\left\langle\eta(x_0,t_0)\eta(x_0+\mathbf{X},t_0+T)\right\rangle$

WAVE GROUP DYNAMICS



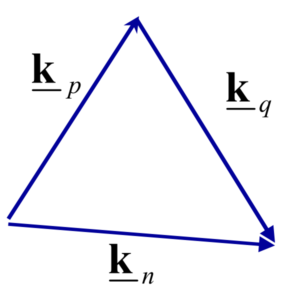
Nonlinear water waves

WEAKLY NONLINEAR ANALYSIS

$$\frac{\partial u}{\partial t} = L(u) + \varepsilon N(u)$$

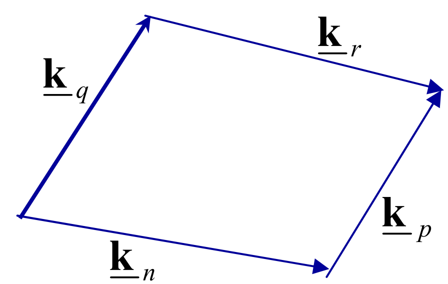
$$u(\underline{\mathbf{x}}, t) = \sum_n a_n(t) \exp\left[i\left(\underline{\mathbf{k}}_n \cdot \underline{\mathbf{x}} + \omega_n t\right)\right]$$

$$\frac{da_n}{dt} + i\omega_n a_n = \varepsilon \sum_{p,q,r} Q_{npq} a_p^* a_q + \varepsilon^2 \sum_{p,q,r} T_{npqr} a_p^* a_q a_r$$



Triad interaction

$$\underline{\mathbf{k}}_n = \underline{\mathbf{k}}_q + \underline{\mathbf{k}}_r$$



Quartet interaction

$$\underline{\mathbf{k}}_n + \underline{\mathbf{k}}_p = \underline{\mathbf{k}}_q + \underline{\mathbf{k}}_r$$

THE ZAKHAROV EQUATION

$$\eta(\underline{\mathbf{x}},t)=\frac{1}{\pi}\sum_n\sqrt{\frac{\omega_n}{2g}}\,B_n(t)\exp\left(\underline{\mathbf{k}}_n\cdot\underline{\mathbf{x}}+\omega_n t\right)+c.c.$$

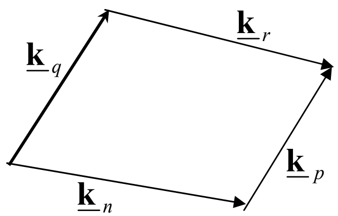
$$\boxed{\frac{dB_n}{dt}+i\omega_n B_n=-i\sum_{p,q,r}T_{npqr}\delta_{n+p-q-r}B_p^*B_qB_r}$$

Conserved quantities : Hamiltonian , wave action and momentum

$$\mathbf{H}=\sum_n\omega_n B_n(t)B_n^*(t)+\frac{1}{2}\sum_{n,p,q,r}T_{npqr}\delta_{n+p-q-r}B_n^*(t)B_p^*(t)B_q(t)B_r(t)$$

$$\mathbf{A}=\sum_n B_n(t)B_n^*(t)$$

$$\mathbf{M}=\sum_n\mathbf{k}_nB_n(t)B_n^*(t)$$



Quartet interaction

$$\underline{\mathbf{k}}_n+\underline{\mathbf{k}}_p=\underline{\mathbf{k}}_q+\underline{\mathbf{k}}_r$$

SUFFICIENT CONDITIONS TO HAVE AN EXTREME CREST

$$\eta(\underline{\mathbf{x}},t)=\frac{1}{\pi}\sum_n\sqrt{\frac{\omega_n}{2g}}\left|B_n(t)\right|\cos\left(\underline{\mathbf{k}}_n\cdot\underline{\mathbf{x}}+\omega_n t+\left|\varphi_n(t)\right|\right)$$

$$B_n(t)=\left|B_n(t)\right|\exp\left[i\varphi_n(t)\right]$$

Set initial conditions

$$B_n(t=-t_0)=\widetilde{B}_n\exp(i\widetilde{\varphi}_n)$$

At (x=0, t=0) we impose that all the harmonic components are in phase (focusing)

$$\varphi_n(0)=0 \qquad n=1,....N$$

$$\frac{dB_n}{dt}+i\omega_n B_n=-i\sum_{p,q,r}T_{npqr}\delta_{n+p-q-r}B_p^*B_qB_r$$

From the ZAKHAROV EQUATION

Amplitude at (x=0,t=0)

$$\nabla\eta=0 \quad and \quad \frac{\partial\eta}{\partial t}=0 \quad at \quad (\underline{\mathbf{x}}=0,t=0)$$

$$H_{\max}=\frac{1}{\pi}\sum_n\sqrt{\frac{\omega_n}{2g}}\left|B_n(0)\right|$$

Stationarity at (x=0,t=0)

SUFFICIENT CONDITIONS TO HAVE AN EXTREME CREST

Maximum amplitude at (x=0,t=0)

$$H_{\max} = \frac{1}{\pi}\sum_n \sqrt{\frac{\omega_n}{2g}}\left|B_n(0)\right|$$

Optimization problem

$$\max \frac{1}{\pi}\sum_n \sqrt{\frac{\omega_n}{2g}}\left|B_n(0)\right|$$

with the following constraints

$$\sum_n \omega_n \left|B_n(0)\right|^2 + \frac{1}{2}\sum_{n,p,q,r} T_{npqr}\,\delta_{n+p-q-r}\left|B_n(0)\right|\left|B_p(0)\right|\left|B_q(0)\right|\left|B_r(0)\right|$$

$$= \sum_n \omega_n \widetilde{B}_n^{\,2} + \frac{1}{2}\sum_{n,p,q,r} T_{npqr}\,\delta_{n+p-q-r}\widetilde{B}_n\widetilde{B}_p\widetilde{B}_q\widetilde{B}_r$$

$$\sum_n \left|B_n(0)\right|^2 = \sum_n \widetilde{B}_n^{\,2} \qquad \sum_n \mathbf{k}_n\left|B_n(0)\right|^2 = \sum_n \mathbf{k}_n\widetilde{B}_n^{\,2}$$

HOW TO CHOOSE THE INITIAL CONDITIONS

Theory of Quasi-Determinism of Boccotti

$$\eta_{\det}(\underline{\mathbf{x}},t)=\frac{H}{\sigma^2}\int E(\underline{\mathbf{k}})\cos(\underline{\mathbf{k}}_n\cdot\underline{\mathbf{x}}-\omega_n t)d\underline{\mathbf{k}} \qquad \frac{H}{\sigma}\to\infty$$

$$\frac{N_{cr}(b,T)}{N_+(b,T)}\to 1 \qquad if \ \frac{H}{\sigma}\to\infty$$

$$\Pr[H>b]=\frac{N_+(b,T)}{N_+(0,T)}=\exp\left(-\frac{b^2}{2\sigma^2}\right) \qquad if \ \frac{b}{\sigma}\to\infty$$

Discrete form

$$\eta_{\det}(\underline{\mathbf{x}},t)=\frac{H}{\sigma^2}\sum_n\frac{1}{2}a_n^2\cos(\underline{\mathbf{k}}_n\cdot\underline{\mathbf{x}}-\omega_n t) \qquad \frac{H}{\sigma}\to\infty$$

Initial conditions which give the highest crest at (x=0,t=0) for linear waves

$$\widetilde{B}_n=\frac{\pi H}{2\sigma^2\sqrt{\omega_n/2g}}a_n^2 \qquad \widetilde{\varphi}_n=0 \qquad n=1,\dots.N$$

THE CONSTRAINED OPTIMIZATION PROBLEM

$$\max_{(X_1,\dots X_N)\in\Re^N} \sum_n w_n X_n \qquad X_n \geq 0$$

$$\sum_n X_n^2 = \sum_n \widetilde{X}_n^{\,2} \qquad \sum_n \mathbf{k}_n X_n^2 = \sum_n \mathbf{k}_n \widetilde{X}_n^{\,2}$$

$$\sum_n w_n X_n^2 + \varepsilon^2 \sum_{n,p,q,r} T_{npqr}\delta_{n+p-q-r} X_n X_p X_q X_r = \sum_n w_n \widetilde{X}_n^2 + \varepsilon^2 \sum_{n,p,q,r} T_{npqr}\delta_{n+p-q-r} \widetilde{X}_n \widetilde{X}_p \widetilde{X}_q \widetilde{X}_r$$

$$H_{\max} = (1+\lambda)H \qquad \frac{H}{\sigma} \to \infty \qquad \lambda = \frac{1}{\pi}\sum \sqrt{w_n} X_n - 1$$

THE EXTREME CREST AMPLITUDE

Third order effects due to nonlinear interaction of free harmonics

$$H_{\max} = (1+\lambda)H \qquad \frac{H}{\sigma} \to \infty \qquad \lambda = \frac{1}{\pi}\sum \sqrt{w_n} X_n - 1$$

Second order effects due to bound harmonics

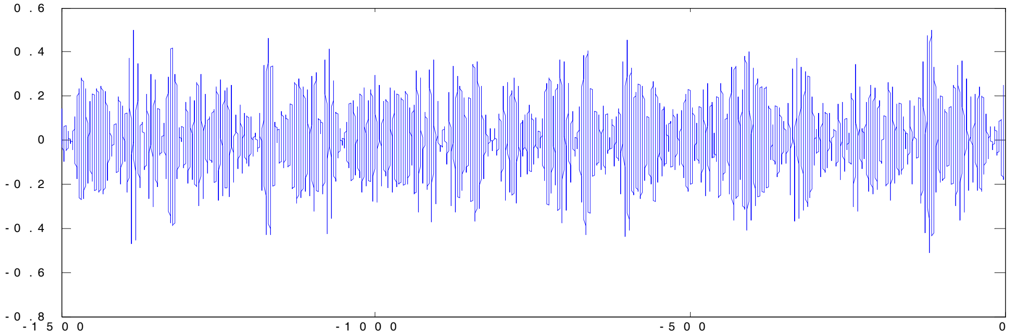
$$h = \sum_n A_n + \frac{1}{4}\sum_{n,s} \Gamma_{ns} A_n A_s$$

$$H_{\max} = (1+\lambda)H + \alpha k_d H^2 \qquad \alpha = \frac{1}{4\pi^2}\sum_{n,s} \Gamma_{ns} \sqrt{w_n w_s} X_n X_s$$

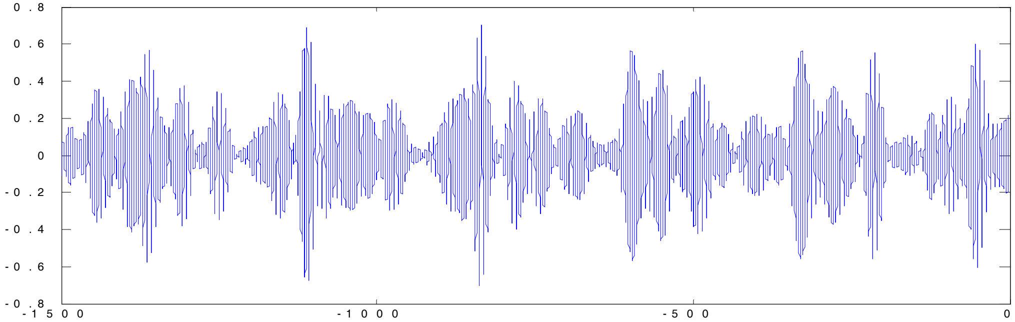
$$\Pr\left(H_{\max} > h\right) = \exp\left[-\frac{(1+\lambda)^2}{8\varepsilon^2\alpha^2}\left(1-\sqrt{1+\frac{4\varepsilon\alpha}{(1+\lambda)^2}\frac{h}{\sigma}}\right)\right]$$

TIME SERIES FROM NUMERICAL SIMULATIONS

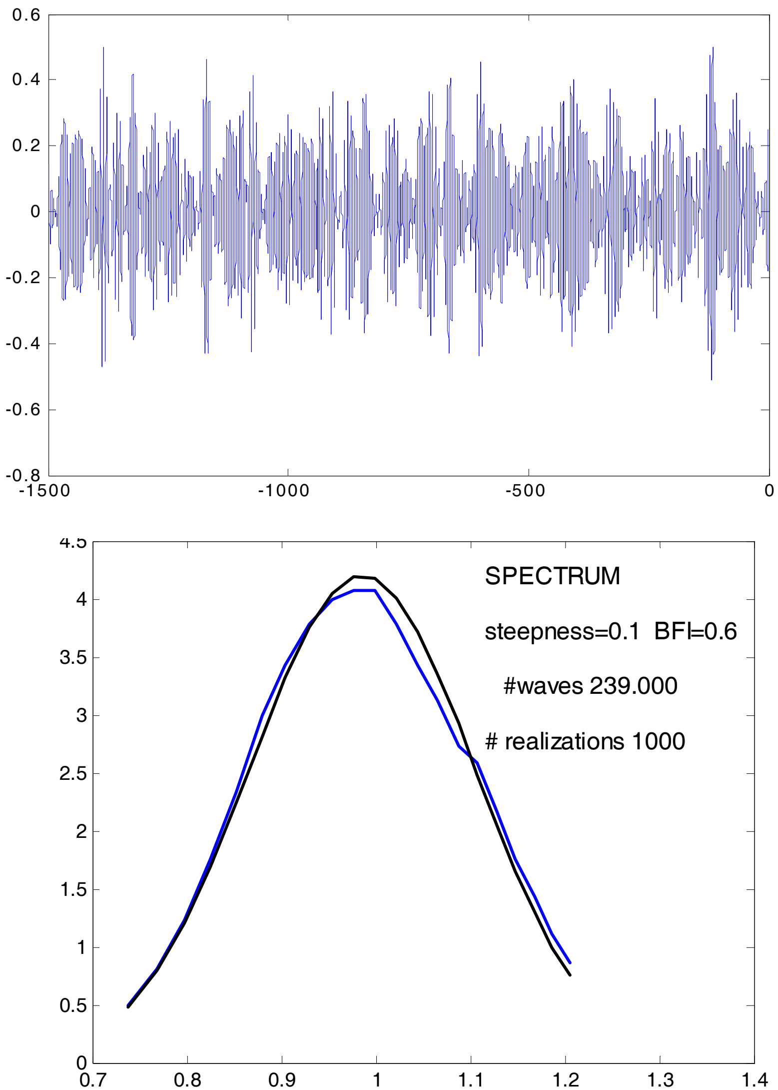
Linear waves

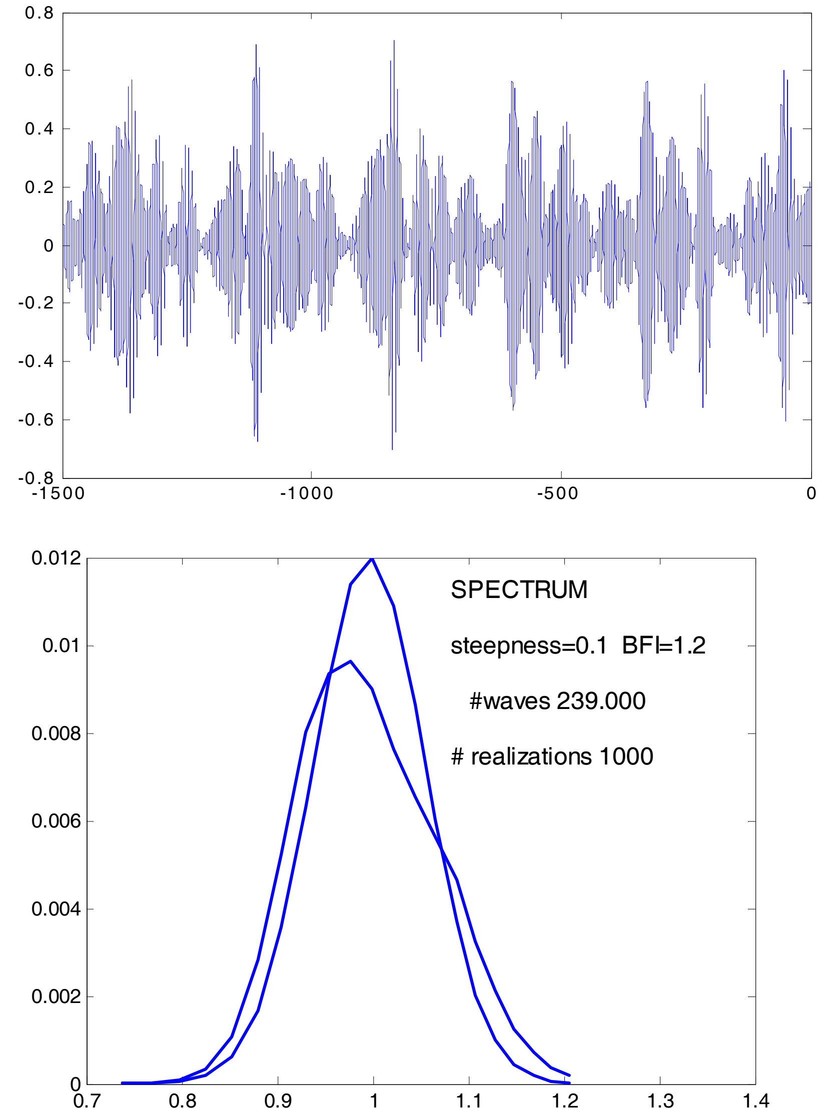


Nonlinear waves

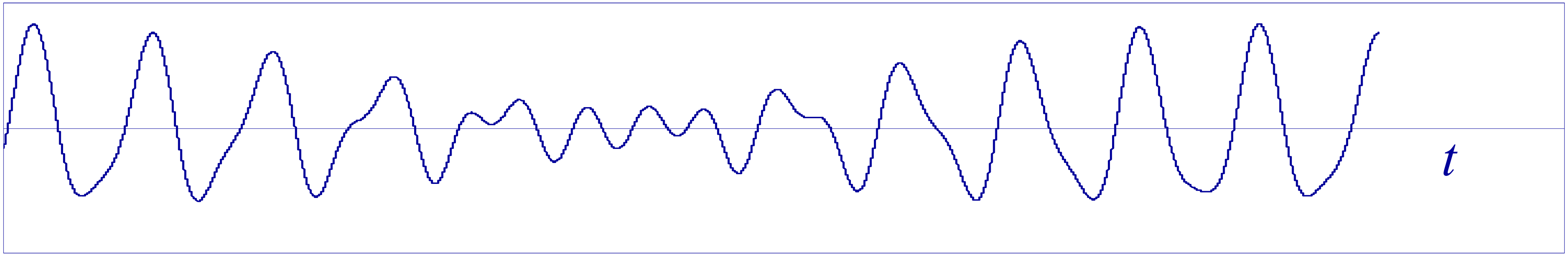


ENERGY SPECTRUM





What is the definition of Probability of exceedance ?



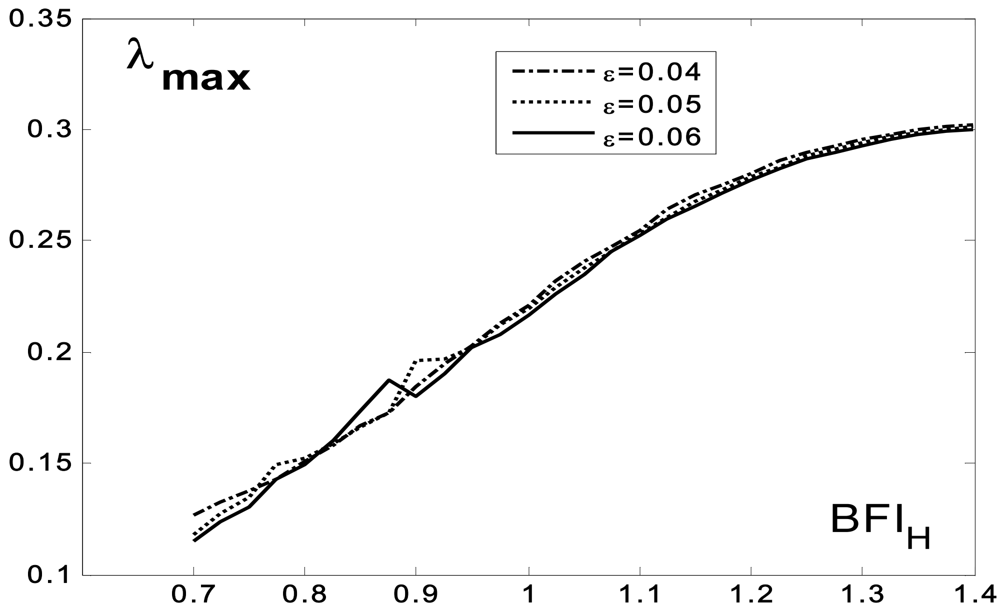
$$P[H] = \frac{\text{number of waves with height greater than } H}{\text{total number of waves}}$$

$$P[Z] = \frac{\text{number of waves with crest greater than } Z}{\text{total number of waves}}$$

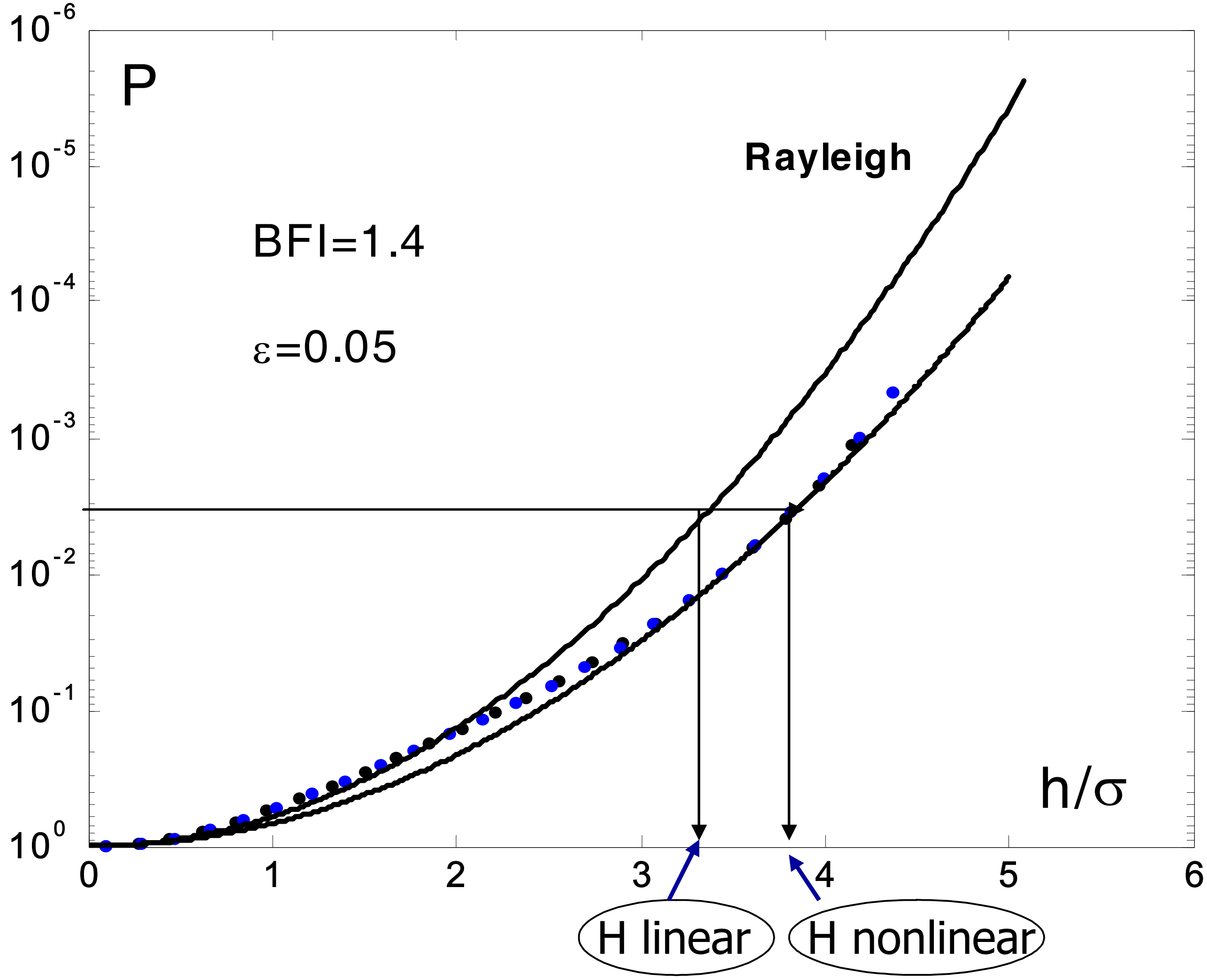
A NEW ANALYTICAL EXPRESSION FOR THE PROBABILITY OF EXCEEDANCE OF A WAVE CREST

$$\Pr\left(H_{\max} > h\right) = \exp\left[-\frac{h^2}{2(1+\lambda)^2\sigma^2}\right]$$

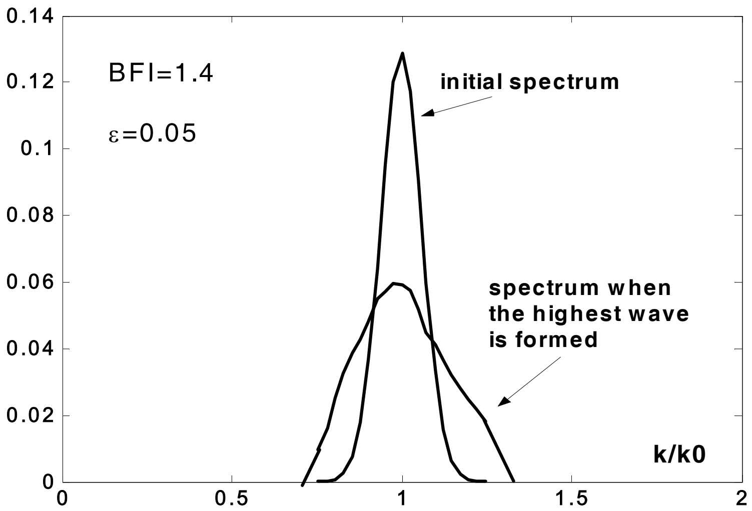
$$\Pr\left(H_{\max} > h\right) = \exp\left[-\frac{(1+\lambda)^2}{8\varepsilon^2\alpha^2}\left(1-\sqrt{1+\frac{4\varepsilon\alpha}{(1+\lambda)^2}\frac{h}{\sigma}}\right)\right]$$

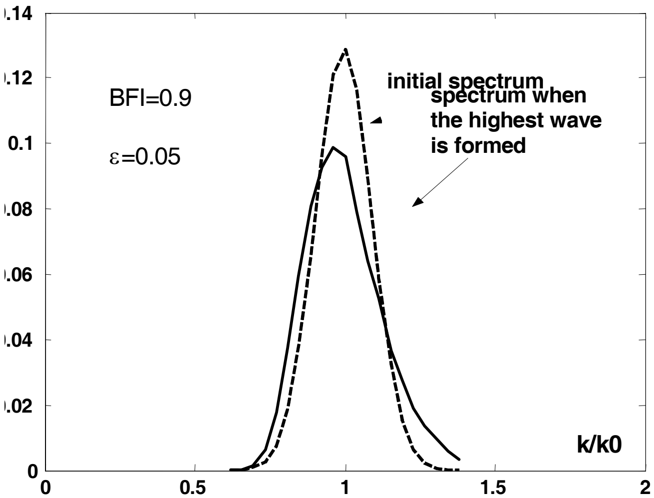


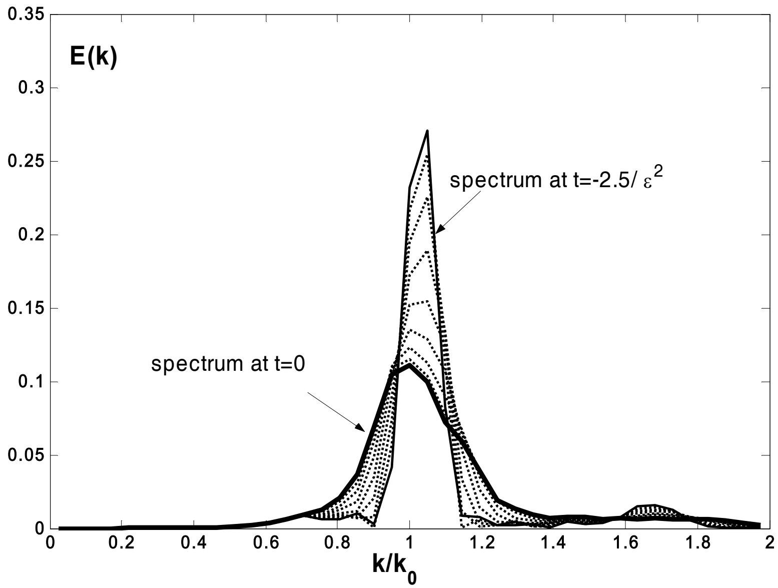
Monte Carlo validation



Evolution of the spectrum







JONSWAP SPECTRA AND DRAUPNER DATA

$$\Pr\left(H_{\max} > h\right) = \exp\left[-\frac{(1+\lambda)^2}{8\varepsilon^2\alpha^2}\left(1-\sqrt{1+\frac{4\varepsilon\alpha}{(1+\lambda)^2}\frac{h}{\sigma}}\right)\right]$$