ON WAVE GROUPS IN A RANDOM SEA

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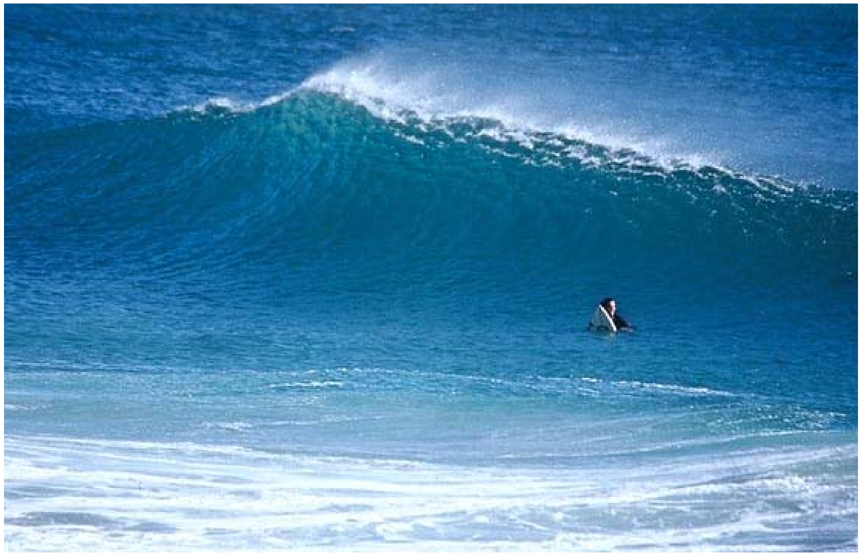
One of the beauty of the nature



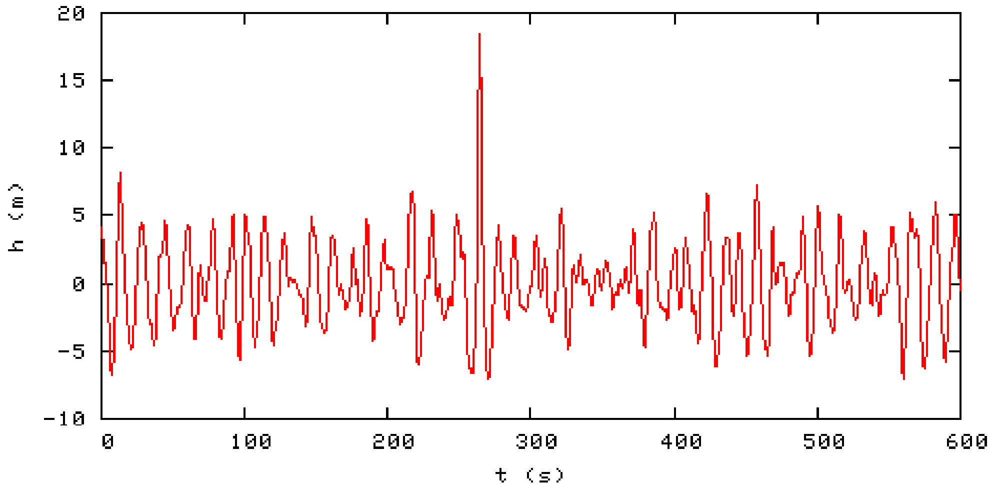
Freak waves

Rogue waves

Giant waves

Extreme waves

DRAUPNER EVENT JANUARY 1995



Freak waves, rogue waves and giant waves

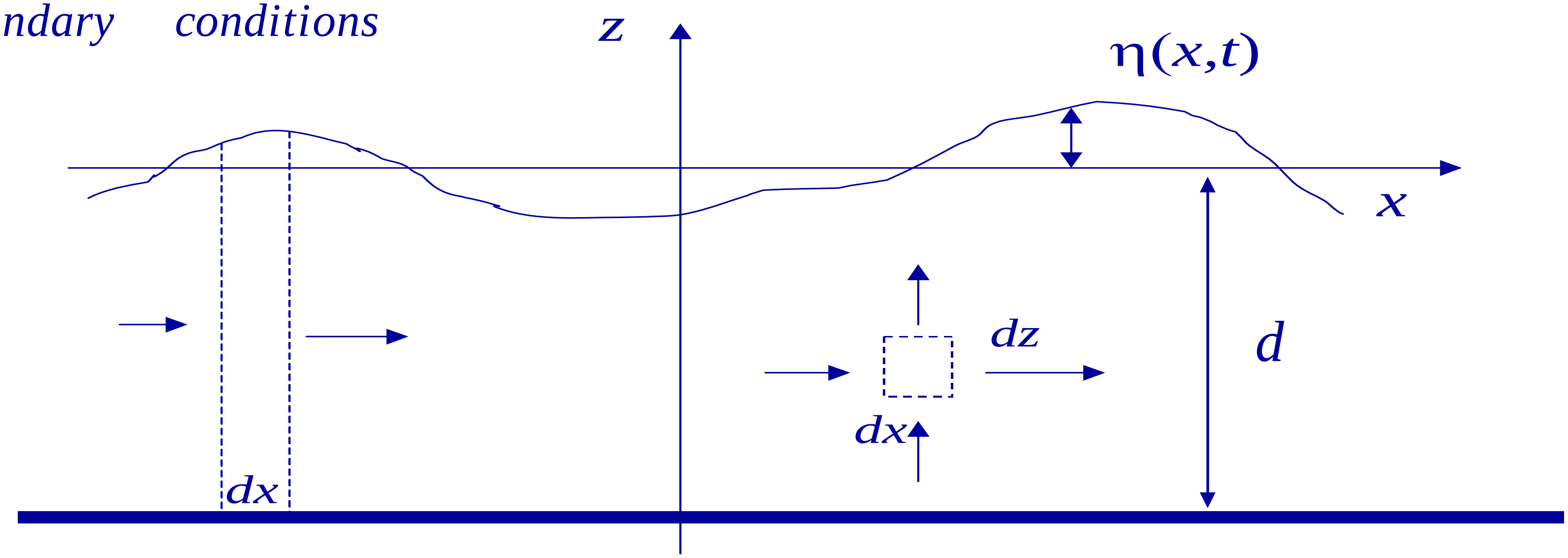
Nonlinear water waves

Gaussian seas
and extreme waves

Stokes Equations for regular waves

$$
\begin{cases}
\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial z^2} = 0 \\[2ex]
\left(\dfrac{\partial \Phi}{\partial z} \right)_{z=\eta} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \eta}{\partial x} \left(\dfrac{\partial \Phi}{\partial x} \right)_{z=\eta} \\[2ex]
\left(\dfrac{\partial \Phi}{\partial t} \right)_{z=\eta} + \dfrac{1}{2} \left[\left(\dfrac{\partial \Phi}{\partial x} \right)^2 + \left(\dfrac{\partial \Phi}{\partial z} \right)^2 \right]_{z=\eta} + g\eta = f(t) \\[2ex]
boundary \quad conditions
\end{cases}
$$

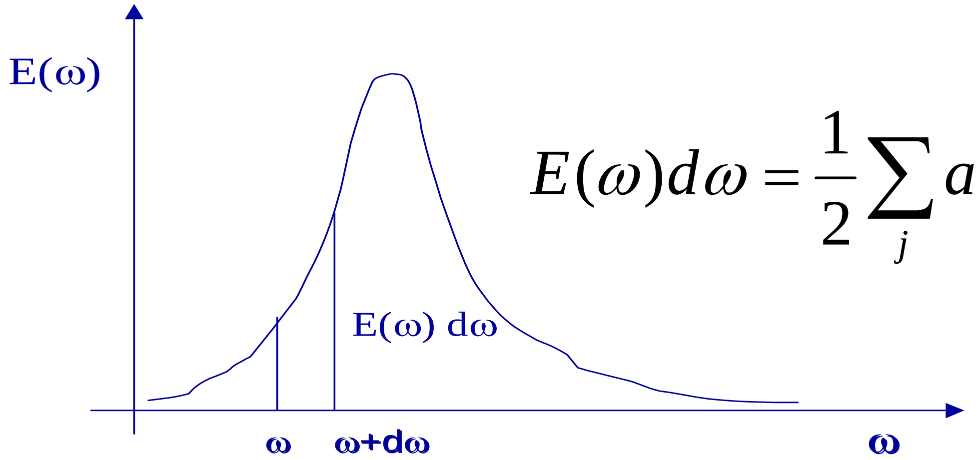
$$v_z = \frac{\partial \Phi}{\partial z} \qquad v_x = \frac{\partial \Phi}{\partial x}$$



Gaussian seas

$$\eta(x,t)=\sum_{j=1}^{N} a_j \cos\left(k_j x+\omega_j t+\varepsilon_j\right)$$

$$\varepsilon_j \quad \textit{uniformly random in } [0,2\pi]$$



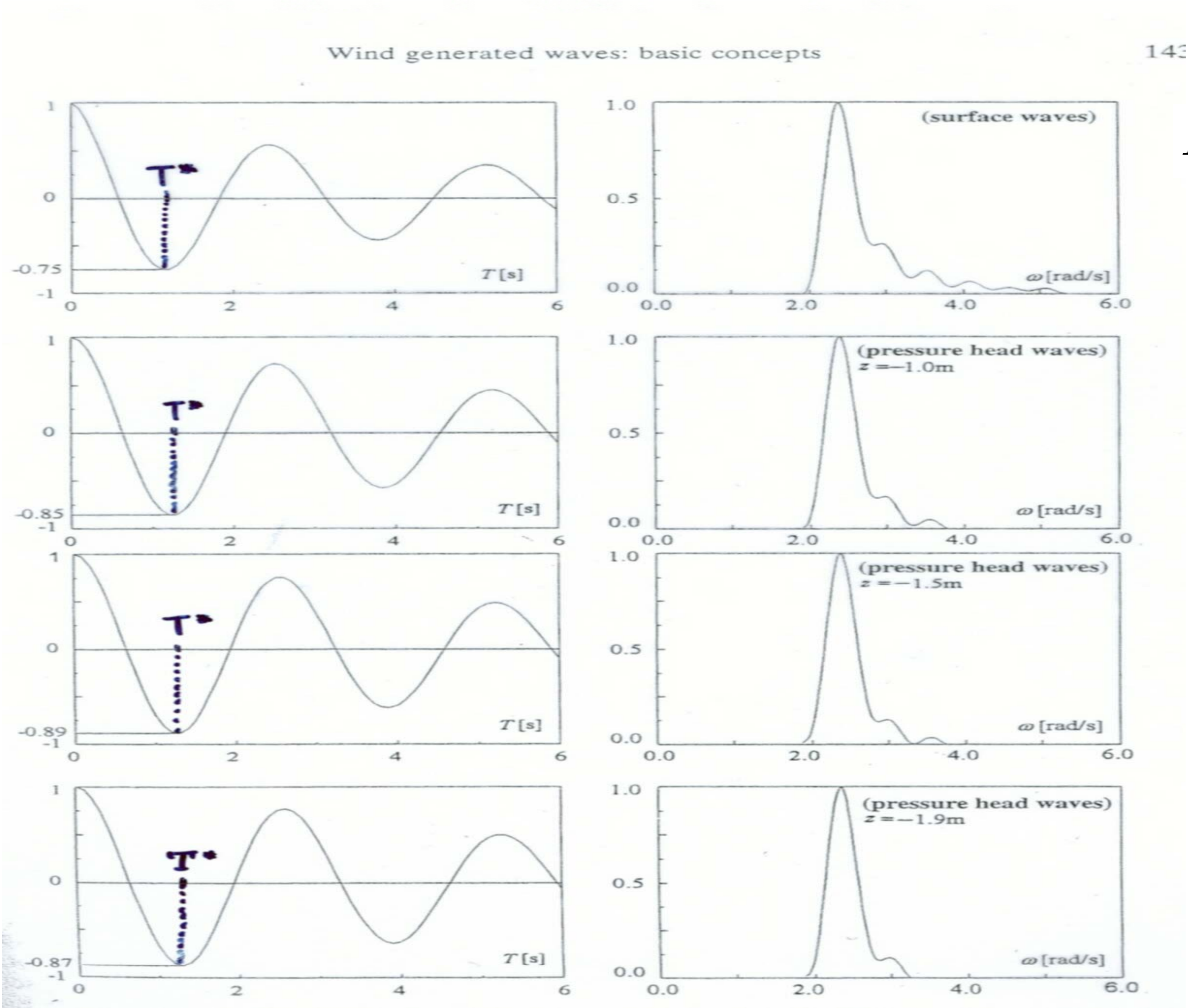
$$E(\omega)d\omega=\frac{1}{2}\sum_j a_j^2 \qquad \omega<\omega_j<\omega+d\omega$$

$$\psi(T)=\langle \eta(x_0,t_0)\eta(x_0,t_0+T)\rangle=\int_0^{\infty} E(\omega)\cos\omega T\, d\omega$$

Stationarity Ergodicity Gaussianity

Typical wave spectra from Mediterranean sea*

$\psi(T)$ $E(\omega)$

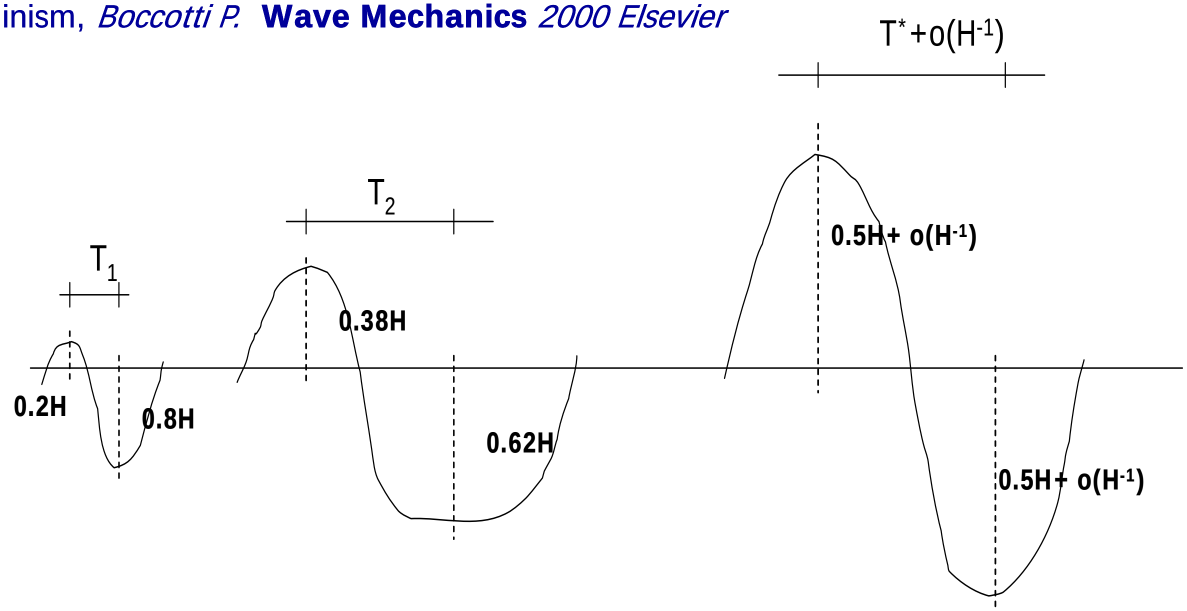


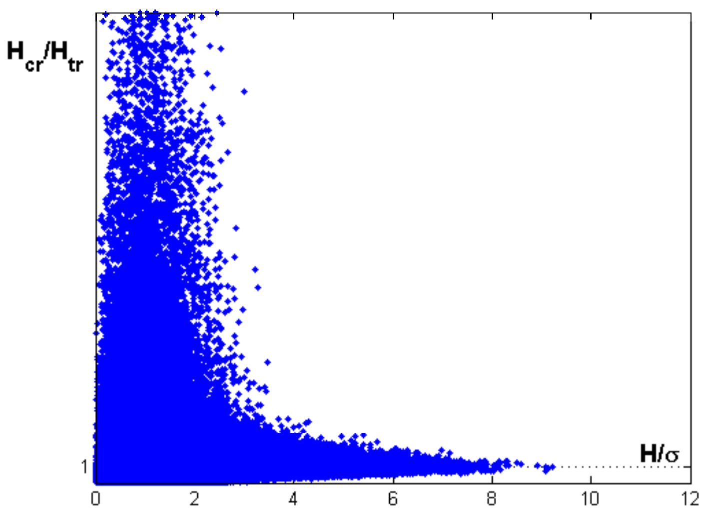
*from *Boccotti P.* **Wave Mechanics** *2000 Elsevier*

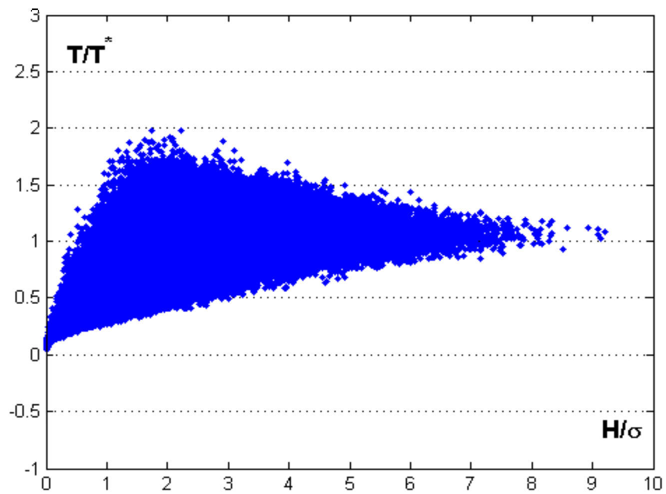
Time domain analysis

NECESSARY AND SUFFICIENT CONDITIONS TO HAVE A HIGH WAVE

*Theory of quasi-determinism, *Boccotti P.* **Wave Mechanics** *2000 Elsevier*







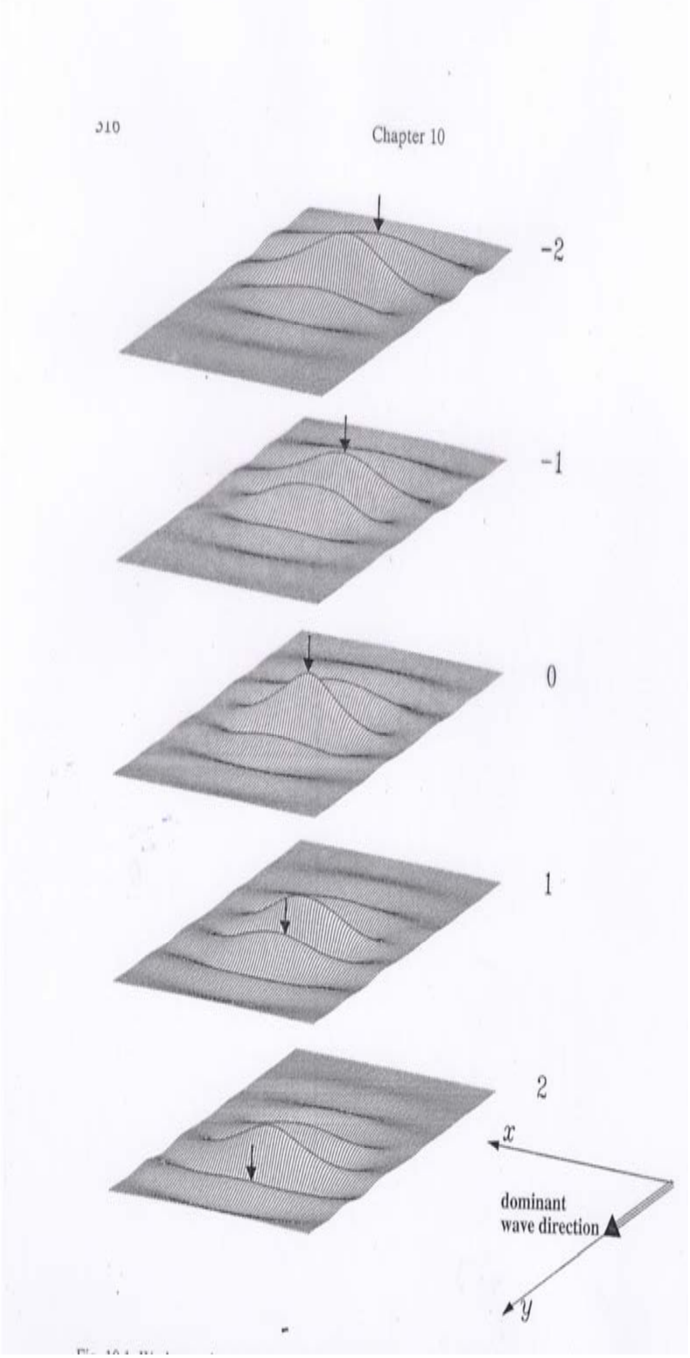
Space-time domain analysis*

What happens in the neighborhood of a point $\mathbf{x}_0$ if a large crest followed by large trough are recorded in time at $\mathbf{x}_0$?

$$\Pr\left[\begin{array}{l} \eta(\mathrm{x}_0+\mathrm{X},t_0+T)\in(u,u+du) \\ \\ conditioned \quad to \\ \eta(\mathrm{x}_0,t_0)=H/2,\ \eta(\mathrm{x}_0,t_0+T_2^*)=-H/2 \end{array}\right]$$

$$\Downarrow \qquad \frac{H}{\sigma}\to\infty$$

$$\delta\left(u-\eta_c\left(\mathbf{x}_0+\mathbf{X},t_0+T\right)\right)$$

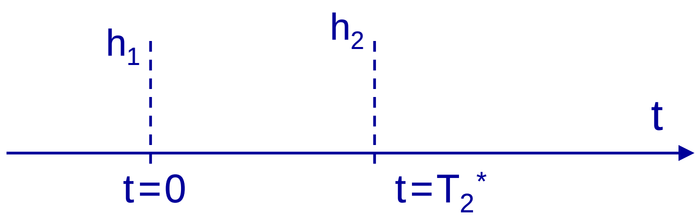


* *Boccotti P.* **Wave Mechanics** *2000 Elsevier*

Time domain analysis: successive wave crests*

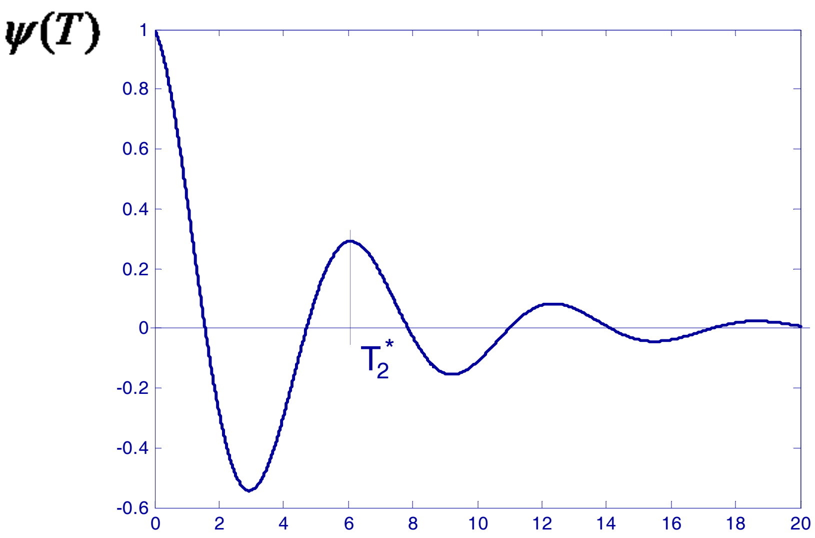
Necessary and sufficient conditions for the occurrence of two high wave crests

$$\eta(t_0) = h_1, \quad \eta(t_0 + T_2^*) = h_2$$

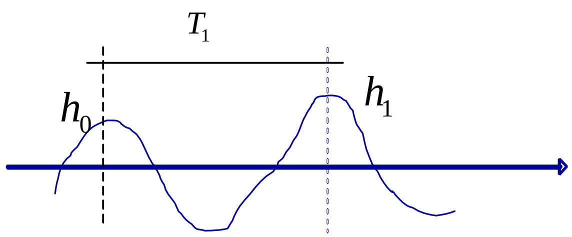


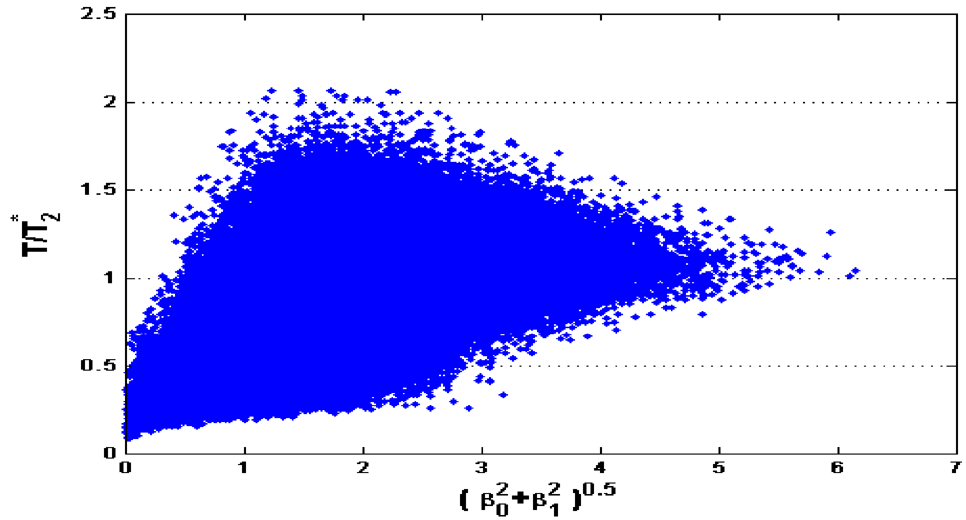
Autocovariance function

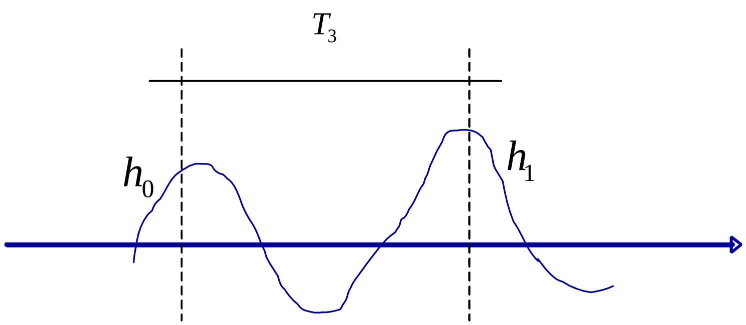
$$\psi(T) = \langle \eta(x_0, t_0)\eta(x_0, t_0 + T) \rangle$$

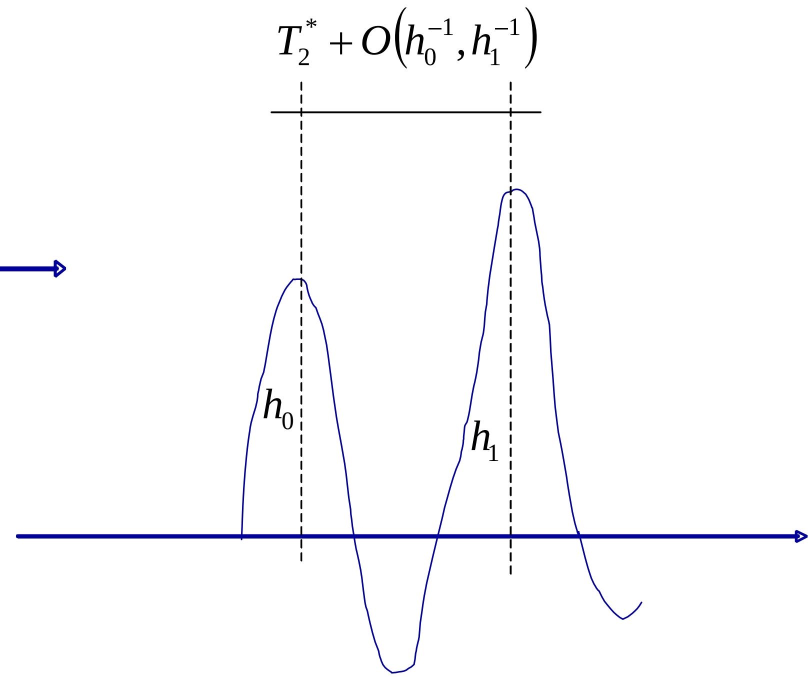


* Fedele F., **Successive wave crests in a Gaussian sea**, *Probabilistic Eng. Mechanics* 2005 (to appear)







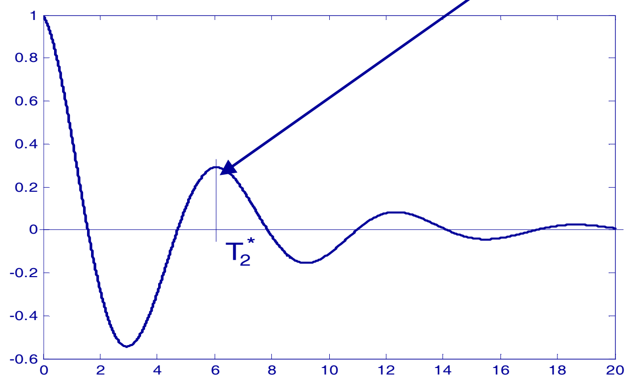


$$as \quad h_0 \to \infty, \quad h_1 \to \infty$$

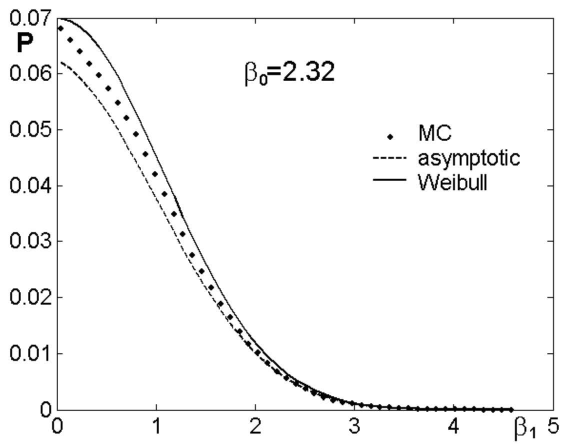
Corollary: joint probability successive wave crests*

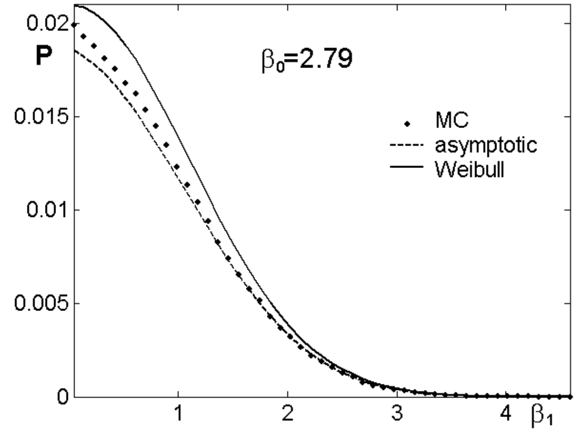
$$p(\beta_0,\beta_1)=\frac{\beta_0\beta_1}{1-k^2}\exp\left[-\frac{\beta_0^2+\beta_1^2}{2(1-k^2)}\right]I_0\left(\frac{k\beta_0\beta_1}{1-k^2}\right)$$

Bivariate Weibull



Monte Carlo Simulations





* Fedele F., **Successive wave crests in a Gaussian sea**, *Probabilistic Eng. Mechanics* 2005 (to appear)

Space-time domain analysis

What happens in the neighborhood of a point $\mathbf{x}_0$
if two large successive wave crests are recorded in time at $\mathbf{x}_0$?

$$\Pr\left[\begin{array}{l}\eta(\mathrm{x}_0+\mathrm{X},t_0+T)\in(u,u+du)\\ \\ \textit{conditioned to}\quad \eta(\mathrm{x}_0,t_0)=h_1,\ \eta(\mathrm{x}_0,t_0+T_2^*)=h_2\end{array}\right]$$

$$\Downarrow \qquad \frac{h_1}{\sigma}\to\infty,\qquad \frac{h_2}{\sigma}\to\infty$$

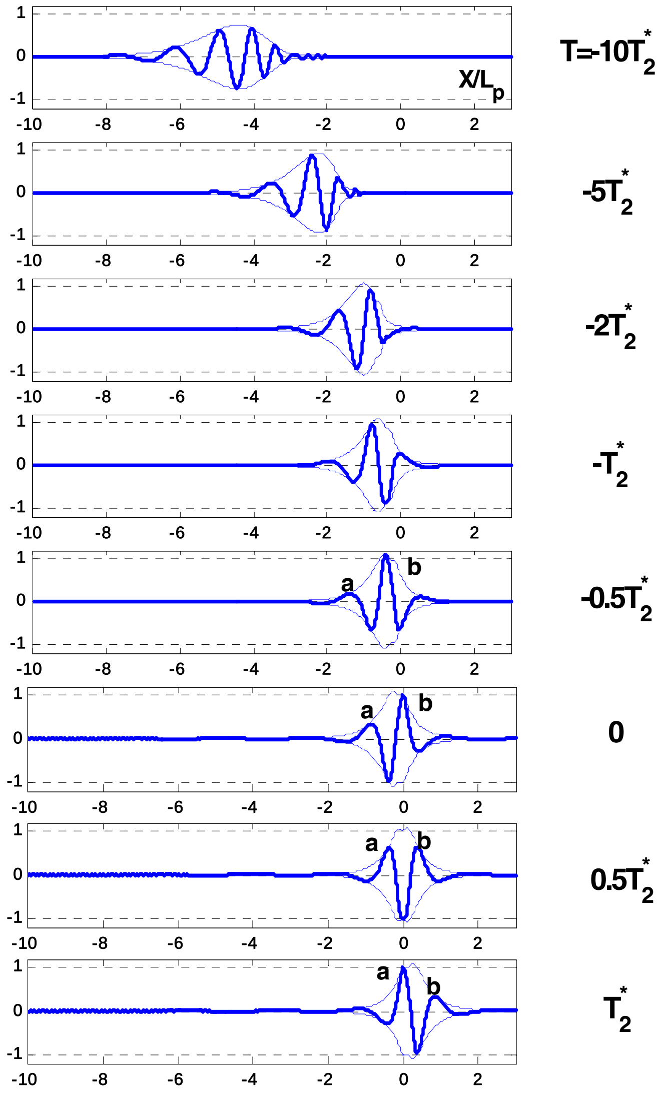
$$\delta\left(u-\eta_c\left(\mathbf{x}_0+\mathbf{X},t_0+T\right)\right)$$

$$\eta_c(\mathbf{X},T)=\frac{\psi(0)h_1-h_2\psi(T_2^*)}{\psi^2(0)-\psi(T_2^*)^2}\,\Psi(\mathbf{X},T)+\frac{\psi(0)h_2-h_1\psi(T_2^*)}{\psi^2(0)-\psi(T_2^*)^2}\,\Psi(\mathbf{X},T-T_2^*)$$

SPACE-TIME covariance

$$\Psi(\mathbf{X},T)=\langle\eta(x_0,t_0)\eta(x_0+\mathbf{X},t_0+T)\rangle$$

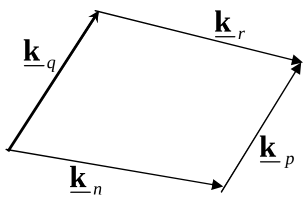
Linear wave group dynamics



Nonlinear evolution of the linear wave group

$$\eta(\underline{\mathbf{x}},t)=\frac{1}{\pi}\sum_n\sqrt{\frac{\omega_n}{2g}}\,B_n(t)\exp\left(i\,\underline{\mathbf{k}}_n\cdot\underline{\mathbf{x}}\right)+c.c.$$

$$\frac{dB_n}{dt}+i\omega_nB_n=-i\sum_{p,q,r}T_{npqr}\,\delta_{n+p-q-r}\,B_p^*B_qB_r$$



Quartet interaction

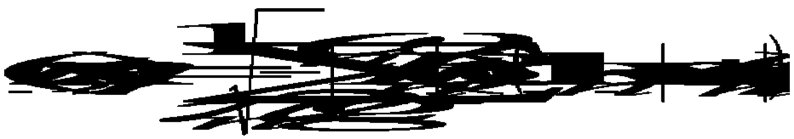
$$\underline{\mathbf{k}}_n+\underline{\mathbf{k}}_p=\underline{\mathbf{k}}_q+\underline{\mathbf{k}}_r$$

Conserved quantities : Hamiltonian , wave action and momentum

$$\mathbf{H}=\sum_n\omega_nB_n(t)B_n^*(t)+\frac{1}{2}\sum_{n,p,q,r}T_{npqr}\,\delta_{n+p-q-r}\,B_n^*(t)B_p^*(t)B_q(t)B_r(t)$$

$$\mathbf{M}=\sum_n\mathbf{k}_nB_n(t)B_n^*(t)\qquad\qquad\mathbf{A}=\sum_nB_n(t)B_n^*(t)$$

Sufficient conditions to have an extreme crest*



At (x=0, t=0) we impose that all the harmonic components are in phase (focusing)

$$\varphi_n(0) = 0 \qquad\qquad n = 1, \ldots. N$$

Constrained optimization problem

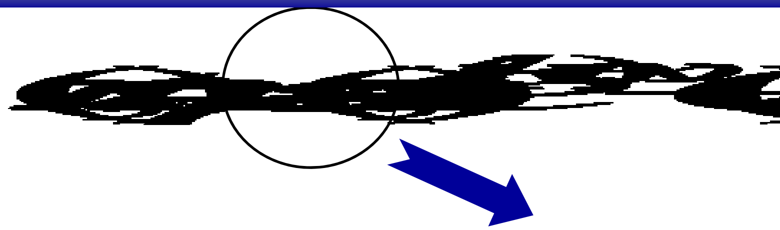
$$H_{NL} = \max \frac{1}{\pi} \sum_n \sqrt{\frac{\omega_n}{2g}} \left| B_n(0) \right|$$

Hamiltonian , wave action and momentum are conserved

$$H_{NL} = (1+\lambda) H_L \qquad\qquad \lambda = \lambda\left(\left\{\left|B_n(0)\right|\right\}\right)$$

*Fedele F. **The occurrence of Extreme crests and the nonlinear wave-wave interaction in random seas**, PROCEEDINGS of XIV International Offshore and Polar Engineering Conference (ISOPE) Toulon, FRANCE may 23-28 2004

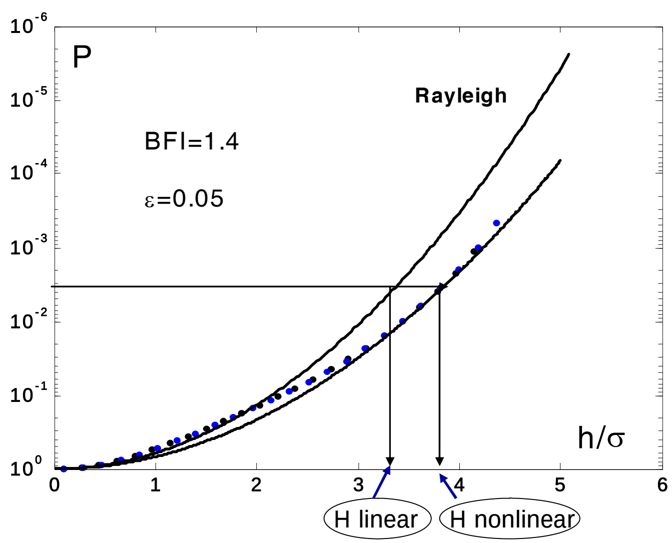
The narrow band limit*
The nonlinear Schrodinger (NLS) equation

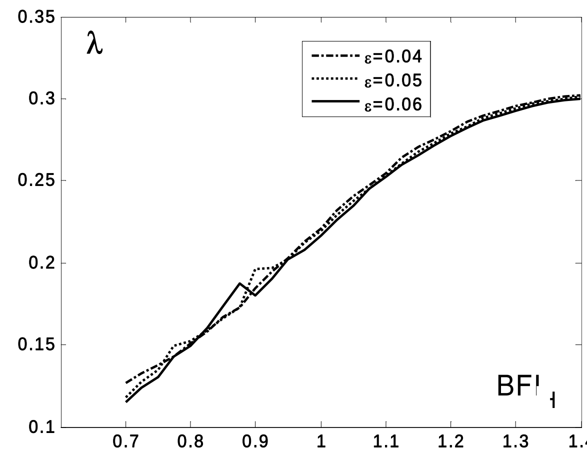


Particular case of the ZAKHAROV EQUATION

$$\frac{\partial A}{\partial t}+i\frac{\partial^2 A}{\partial x^2}+iBFI^2|A|^2A=0$$

$$\Pr(H_{max}>h)=\exp\left[-\frac{h^2}{2(1+\lambda)^2\sigma^2}\right]$$





Intermittency (FERMI-ULAM PASTA recurrence)

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Recurrent solutions* & Intermittency

$$\text{Rayleigh quotient} \quad Q(t) = \frac{\int \left| \frac{\partial A}{\partial x} \right|^2 dx}{\int |A|^2 dx} \le const \qquad \text{bounded in time}$$

[illegible]

SPECTRAL BANDWIDTH

$$Q(t) = \frac{\sum_n k_n^2 |a_n(t)|^2}{\sum_n |a_n(t)|^2} \approx \Delta K^2(t) \quad \text{bounded in time}$$

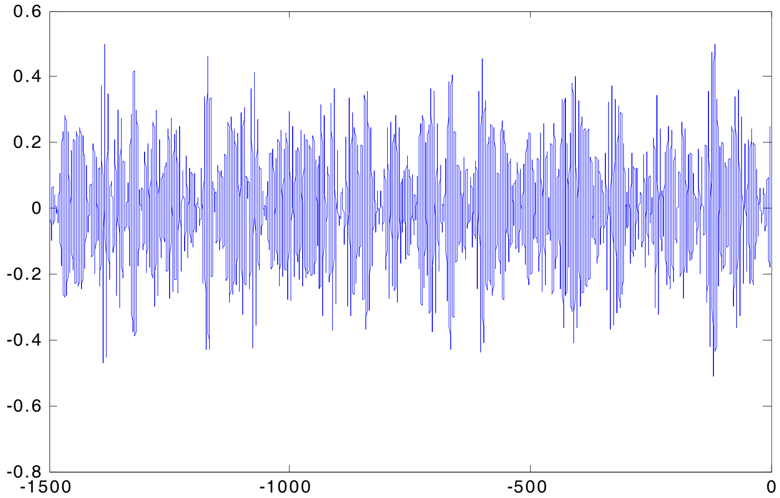
IF ENERGY CONTENT OF SMALL SCALES IS SMALL
IT REMAINS SMALL FOR ALL THE TIME

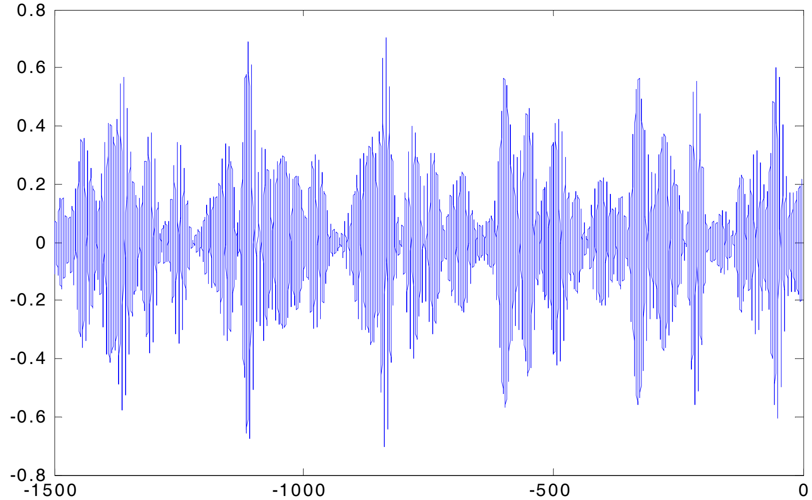
ENERGY PERPETUALLY DISTRIBUTED BETWEEN
FINITE SET OF MODES → RECURRENT SOLUTIONS

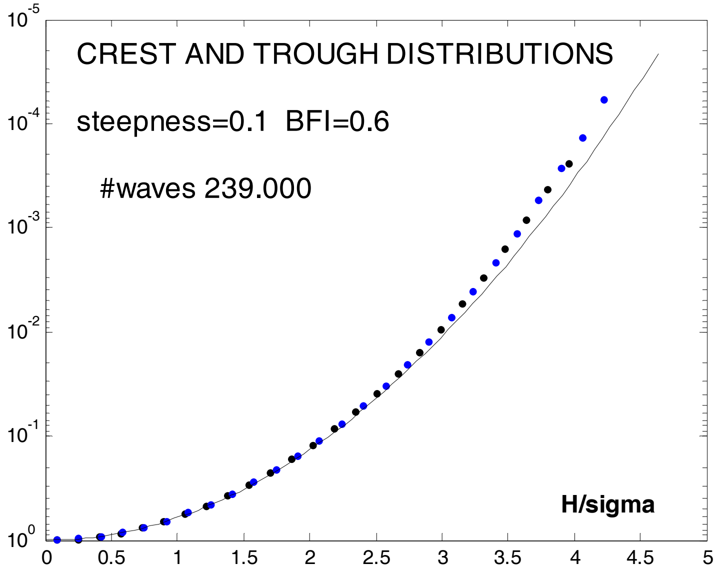
BENJAMIN-FEIR INSTABILITY & FERMI-ULAM PASTA RECURRENCE

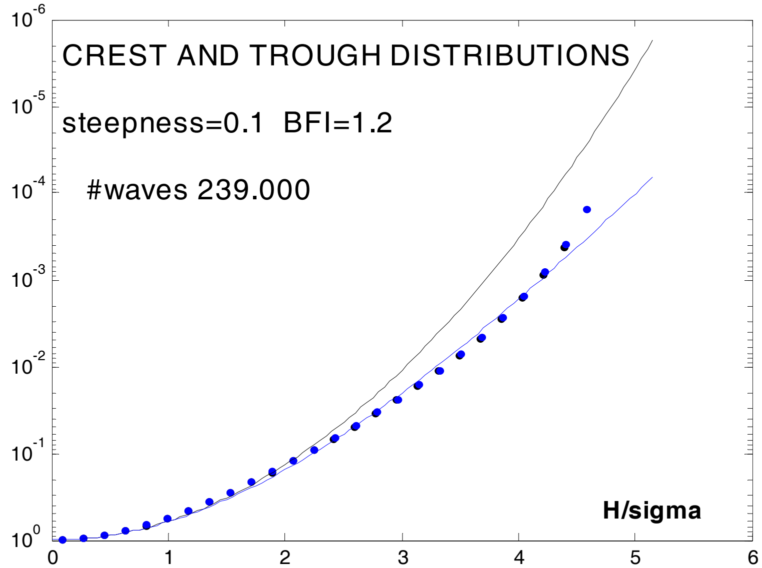
*Thyagaraja. **Recurrent motions in certain continuum dynamical systems** Physics of Fluids 22(11) 1979, pp. 2093-2096

Monte-Carlo simulations





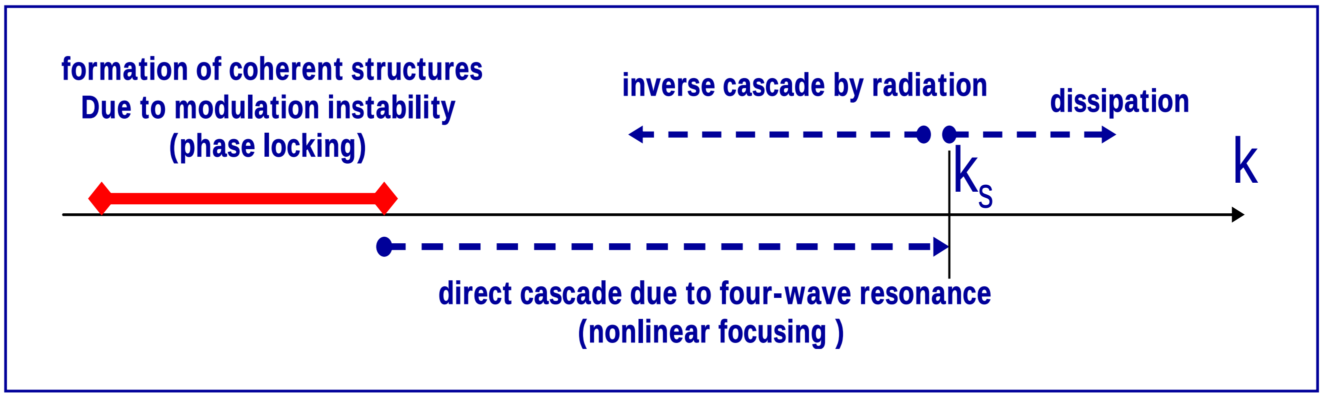


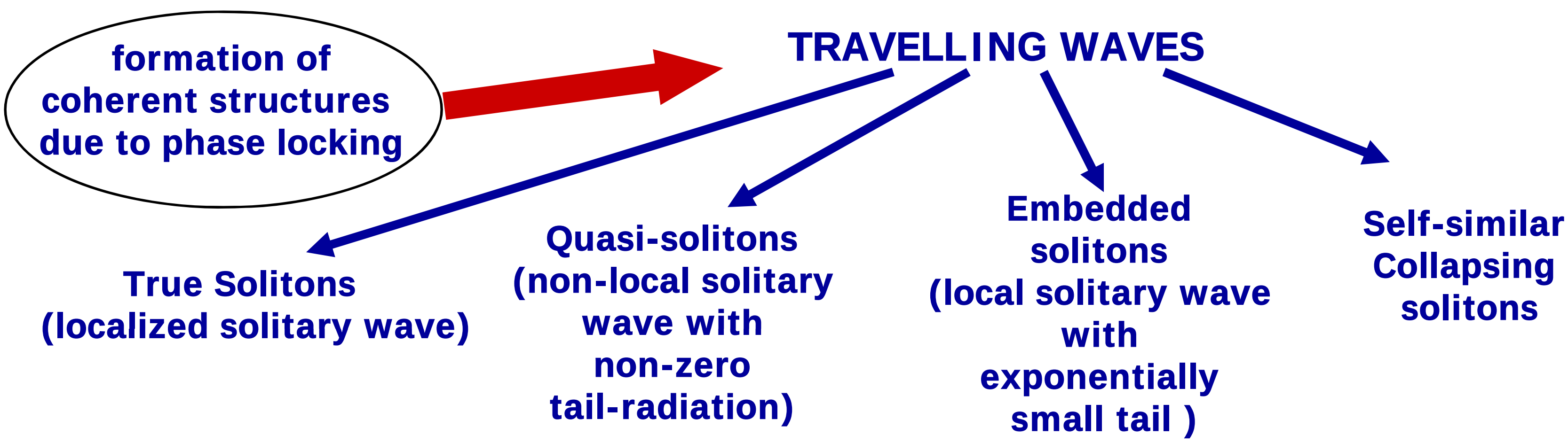


Quasi-solitonic wave turbulence*

$$i\frac{\partial \hat{a}_k}{\partial t} = \omega(k)\hat{a}_k + \iiint T_{123k}\hat{a}_1\hat{a}_2\hat{a}_3^*\delta(k_1+k_2-k_3-k)dk_1dk_2dk_3$$

$$A(x,t) = \int \hat{a}_k(t)\exp(i\,k\,x)dk$$





*Cai, Majda, Laughlin & Tabak, **Dispersive wave turbulence in one dimension** PHYSICA D 152-153 (2001) 551-572
Zakharov, Dias & Pushkarev, **One-dimensional wave turbulence** Physics reports 398 (2004) 1-65