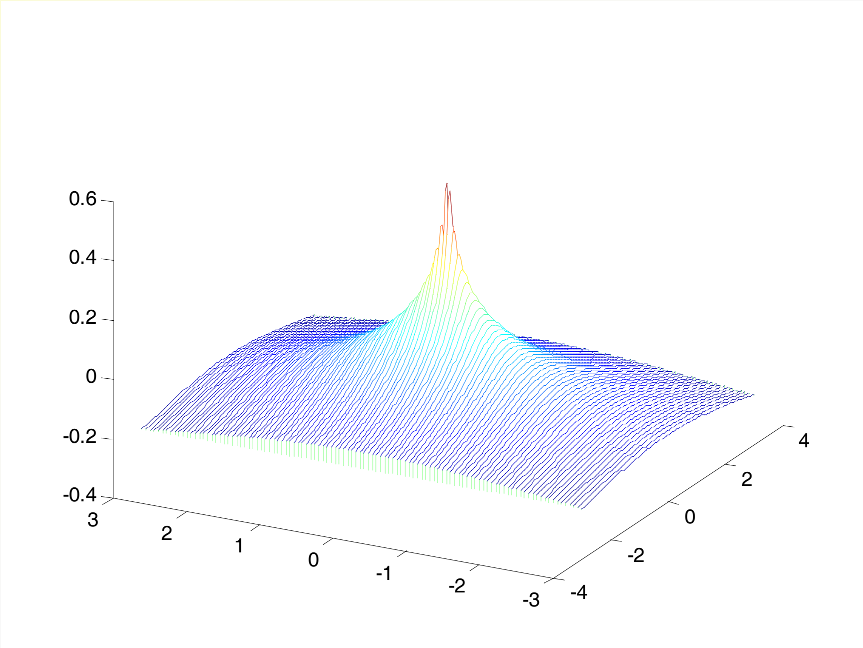
A BOUNDARY ELEMENT METHOD

FOR FLUORESCENCE TOMOGRAPHY



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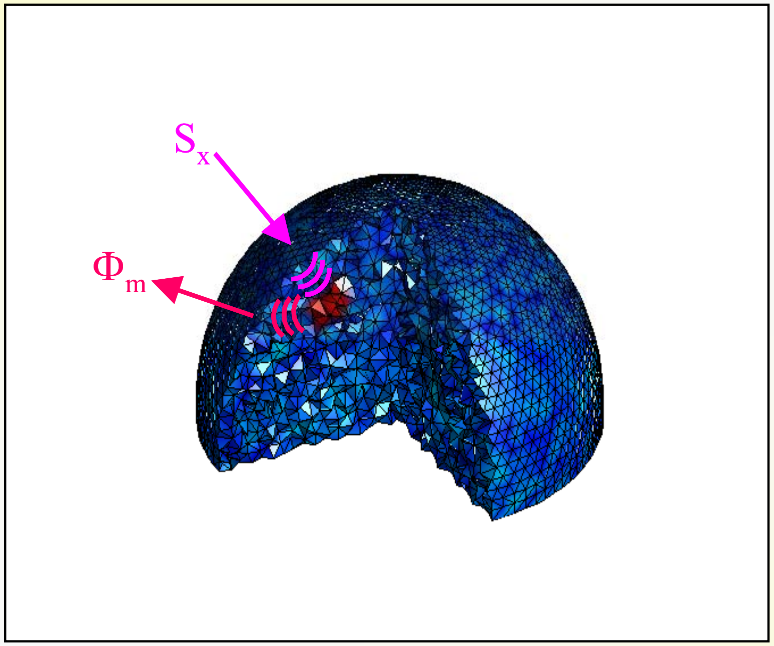
FREQUENCY-DOMAIN PHOTON MIGRATION PDE's

$$-\nabla \bullet \left(D_x \nabla \Phi_x \right) + k_x \Phi_x = S_x$$

$$-\nabla \bullet \left(D_m \nabla \Phi_m \right) + k_m \Phi_m = \beta \Phi_{\mathrm{x}}$$



$$\begin{cases} D_x \dfrac{\partial \Phi_x}{\partial n} + r_x \Phi_x = 0 \\ \\ D_m \dfrac{\partial \Phi_m}{\partial n} + r_m \Phi_m = 0 \end{cases} \quad on \quad \partial\Omega$$

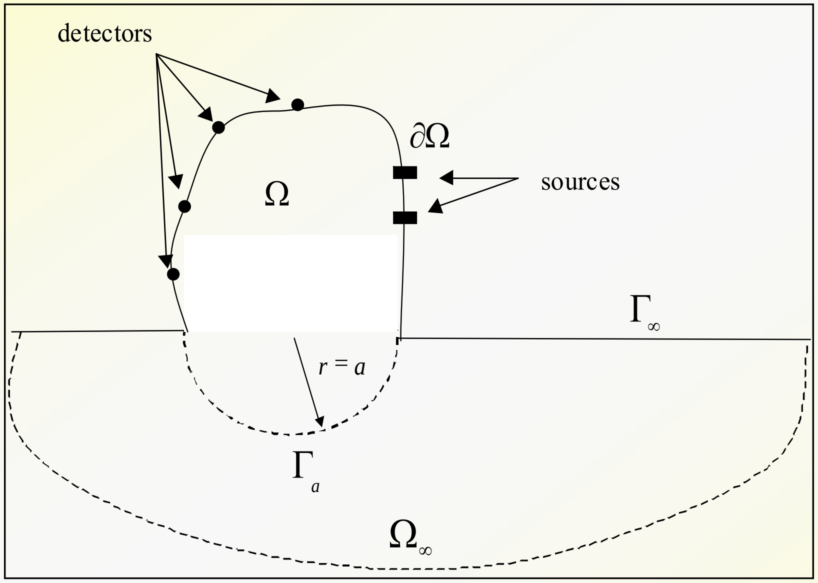
THE INVERSE PROBLEM

$$L\Phi(x,\mu) = f(\mu) \quad L \;\; diffusion - decay \;\; operator$$

$$F(\mu) = \sum_{n=1}^{N} \left(\Phi_n^{\det} - \Phi_{meas}^{\det} \right)^2$$

$$find \;\; \mu \;\; such \;\; that \;\; F(\mu) \; \min$$

Sensitivity



$$\frac{\delta F}{\delta \mu} = 2\sum_{n=1}^{N} \left(\Phi_n^{\det} - \Phi_{meas}^{\det} \right) \frac{\delta \Phi_n^{\det}}{\delta \mu} = 0$$

$$\sum_{n=1}^{N} \left(\overline{\Phi}_n^{\det} + \frac{\delta \Phi_n^{\det}}{\delta \mu} \delta\mu - \Phi_{meas}^{\det} \right) \frac{\delta \Phi_n^{\det}}{\delta \mu} = 0$$

GENERALIZED FOURIER EXPANSION IN SPHERICAL COORDINATES

$$-\nabla\bullet\left(D_x\nabla\Phi_x\right)+k_x\Phi_x=S_x$$

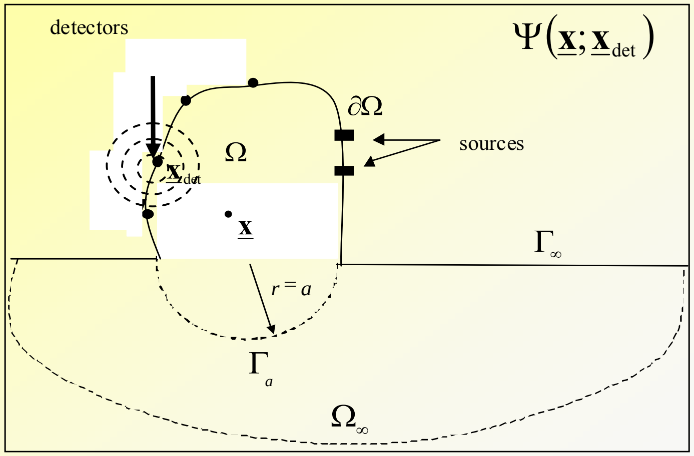
$$-\nabla\bullet\left(D_m\nabla\Phi_m\right)+k_m\Phi_m=\beta\,\Phi_x$$

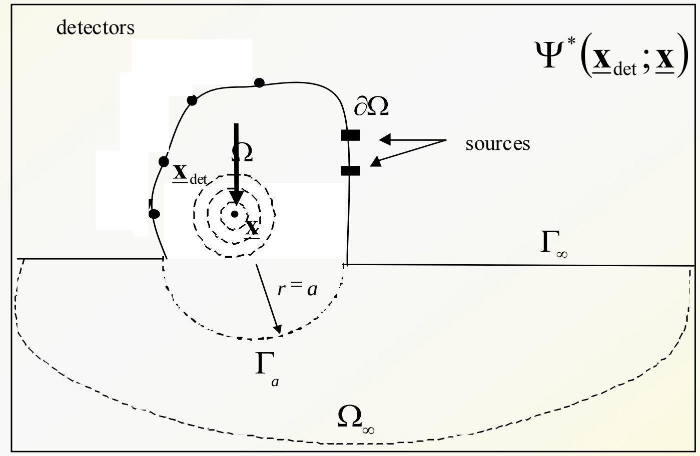
$$\Phi_x(r,\theta,\phi)=\sum A_{nm}\exp\left(im\theta\right)P_n^m(\phi)\frac{J_{n+1/2}(\sqrt{-k_x/D_x}\,r)}{\sqrt{r}}$$

$$\Phi_m(r,\theta,\phi)=$$

$$\sum\exp\left(im\theta\right)P_n^m(\phi)\left[B_{nm}\frac{J_{n+1/2}(\sqrt{-k_m/D_m}\,r)}{\sqrt{r}}-\frac{\beta/D_m}{k_x/D_x-k_m/D_m}A_{nm}\frac{J_{n+1/2}(\sqrt{-k_x/D_x}\,r)}{\sqrt{r}}\right]$$

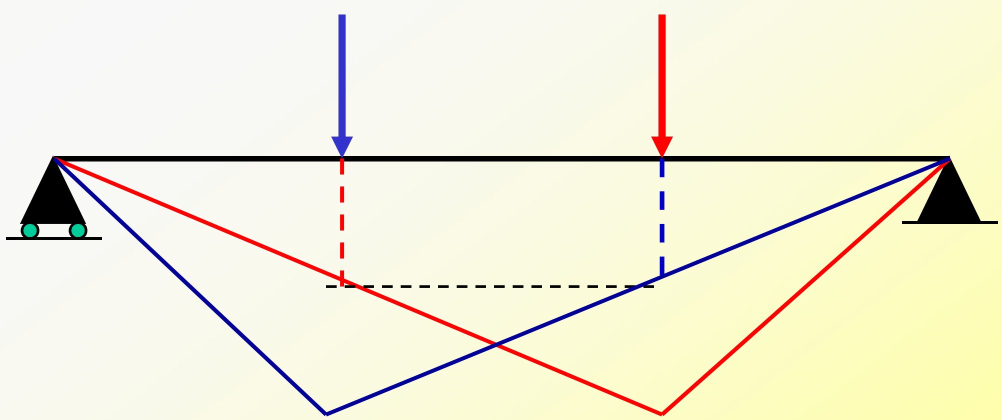
GREEN'S FUNCTION AND THE FUNDAMENTAL SOLUTION FOR SELF-ADJOINT OPERATORS





$$L(\Psi) = \delta(\underline{x} - \underline{x}_{\det})$$

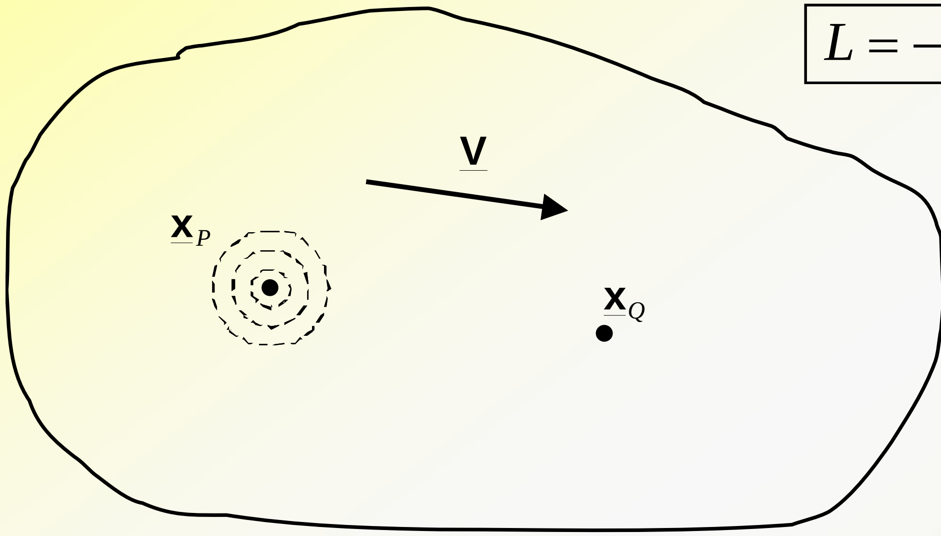
$$L^*(\Psi) = \delta(\underline{x} - \underline{x}_{\det})$$



NON SELF-ADJOINT OPERATORS

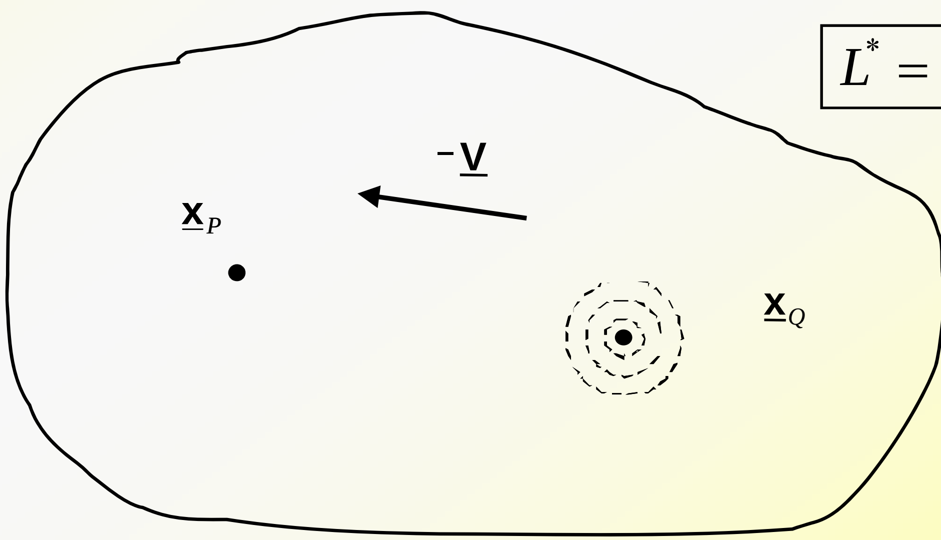
Fundamental solution $\Psi^*(\underline{\mathbf{x}}_Q, \underline{\mathbf{x}}_P)$

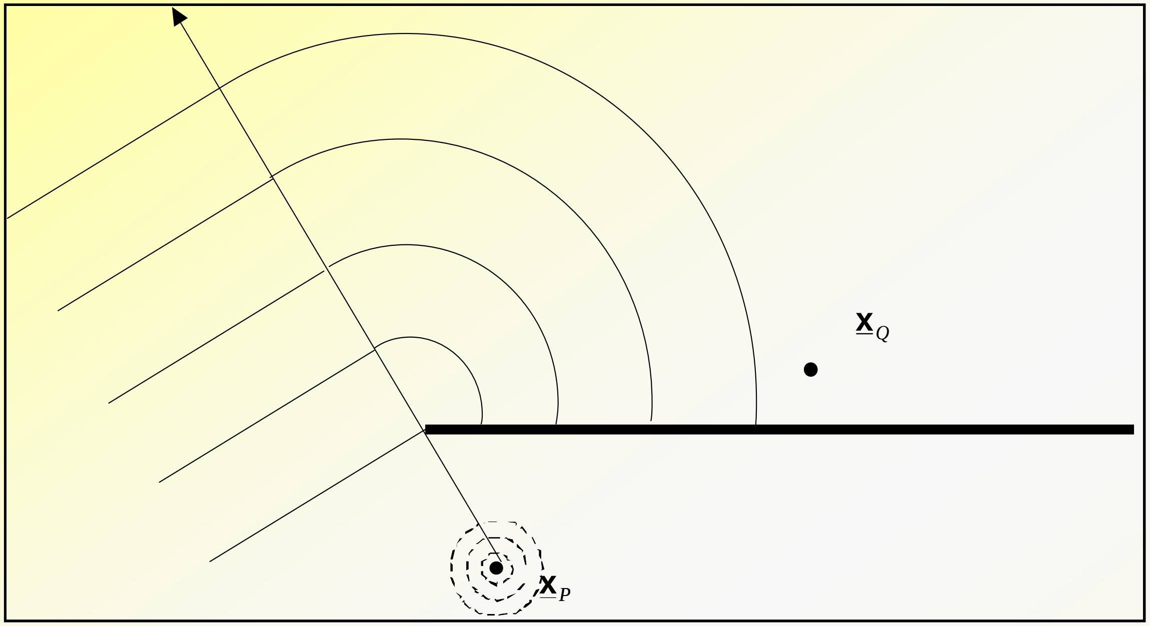
$$L = -\nabla \bullet (D\nabla\Phi) + \underline{V} \bullet \nabla\Phi$$

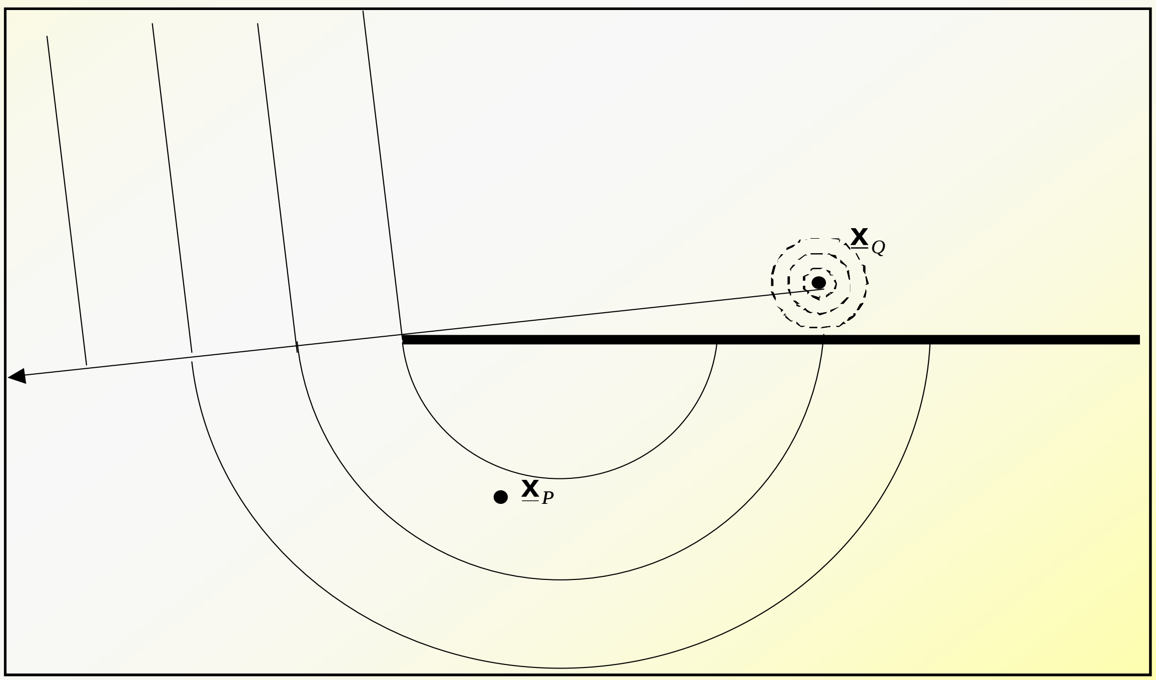


Adjoint solution (Green function)

$$L^* = -\nabla \bullet (D\nabla\Phi) - \underline{V} \bullet \nabla\Phi$$




$\underline{\mathbf{x}}_Q$
$\underline{\mathbf{x}}_P$


$\underline{\mathbf{x}}_Q$
$\underline{\mathbf{x}}_P$

BEM FOR LAPLACE EQUATION

$$\nabla^2 \Phi(\underline{x}) = 0 \quad \Rightarrow \quad -\int_{\Omega} \boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) \nabla^2 \boldsymbol{\Phi}(\underline{\boldsymbol{x}})\, \boldsymbol{d}\,\underline{\boldsymbol{x}} = 0$$

Integration by parts twice

$$\int_{\Omega} -\nabla^2 \boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) \boldsymbol{\Phi}(\underline{\boldsymbol{x}})\, \boldsymbol{d}\,\underline{\boldsymbol{x}} + \int_{\partial\Omega} \left(-\boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) \frac{\partial \Phi(\underline{\boldsymbol{x}})}{\partial \boldsymbol{n}} + \frac{\partial \boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0)}{\partial \boldsymbol{n}} \Phi(\underline{\boldsymbol{x}}) \right) \boldsymbol{dS} = 0$$

$$\int_{\Omega} \delta(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) \boldsymbol{\Phi}(\underline{\boldsymbol{x}})\, \boldsymbol{d}\,\underline{\boldsymbol{x}} + \int_{\partial\Omega} \left(-\boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) \frac{\partial \Phi(\underline{\boldsymbol{x}})}{\partial \boldsymbol{n}} + \frac{\partial \boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0)}{\partial \boldsymbol{n}} \Phi(\underline{\boldsymbol{x}}) \right) \boldsymbol{dS} = 0$$

$$\frac{1}{2} \boldsymbol{\Phi}(\underline{\boldsymbol{x}}_0) + \int_{\partial\Omega} \left(-\boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) \frac{\partial \Phi(\underline{\boldsymbol{x}})}{\partial \boldsymbol{n}} + \frac{\partial \boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0)}{\partial \boldsymbol{n}} \Phi(\underline{\boldsymbol{x}}) \right) \boldsymbol{dS} = 0$$

$$\nabla^2 \boldsymbol{G}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) + \delta(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) = 0$$

Green's function

DISCRETIZATION

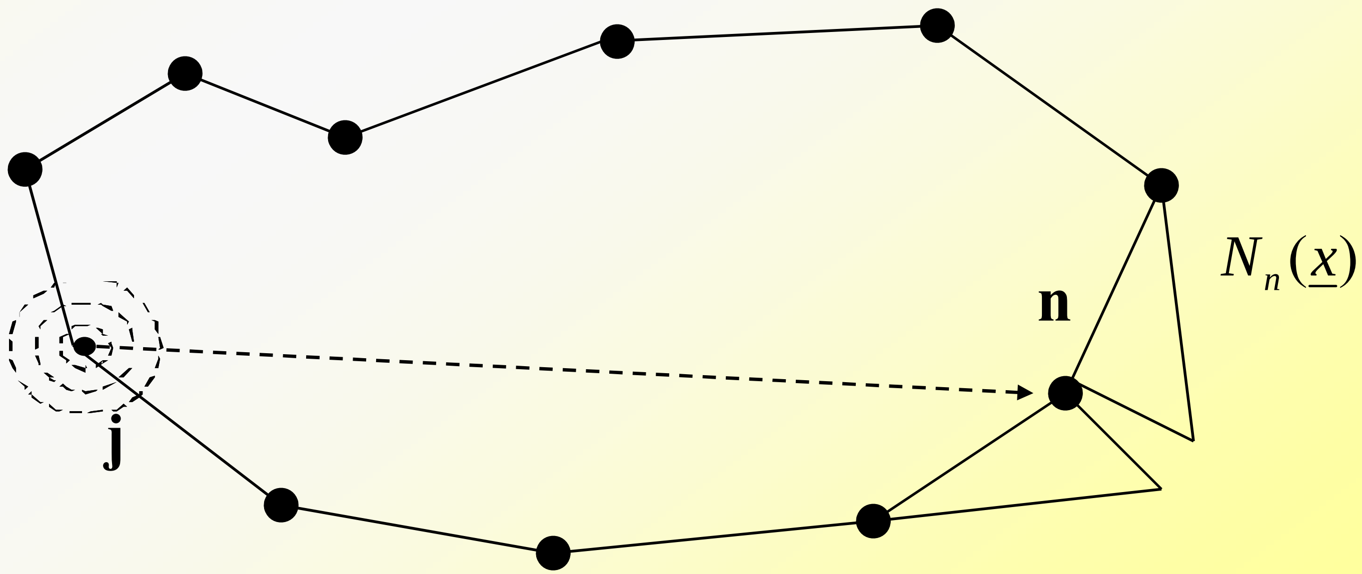
$$\frac{1}{2}\boldsymbol{\Phi}(\underline{x}_0) + \int_{\partial\Omega}\left(-G(\underline{x},\underline{x}_0)\frac{\partial\Phi(\underline{x})}{\partial n} + \frac{\partial G(\underline{x},\underline{x}_0)}{\partial n}\Phi(\underline{x})\right)dS = 0$$

$$\boldsymbol{\Phi}(\underline{x}) \cong \sum_n \phi_n N_n(\underline{x})$$

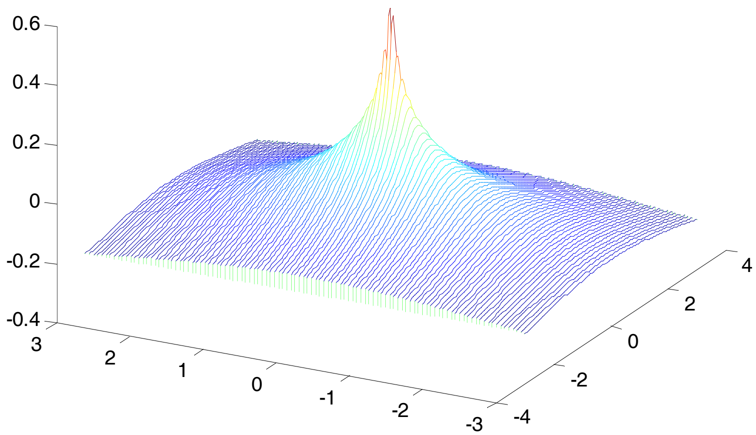
$$\frac{\partial\Phi(\underline{x})}{\partial n} \cong \sum_n Q_n N_n(\underline{x})$$

$$\sum_n\left[\frac{1}{2}\delta_{jn} + \int_{\partial\Omega}\frac{\partial G(\underline{x},\underline{x}_j)}{\partial n}N_n(\underline{x})\,d\Gamma\right]\phi_n - \sum_n\left[\int_{\partial\Omega}G(\underline{x},\underline{x}_j)N_n(\underline{x})\,d\Gamma\right]Q_n = 0$$

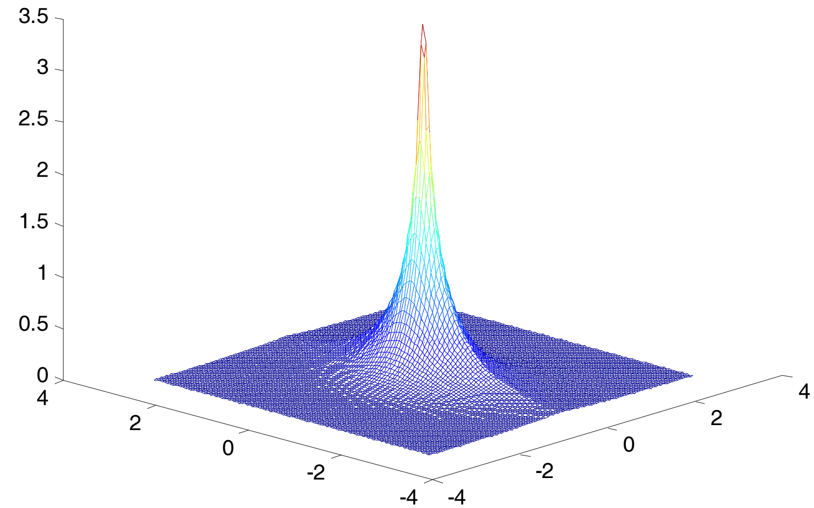
IN GENERAL NON SYMMETRIC MATRICES (only for special boundaries)



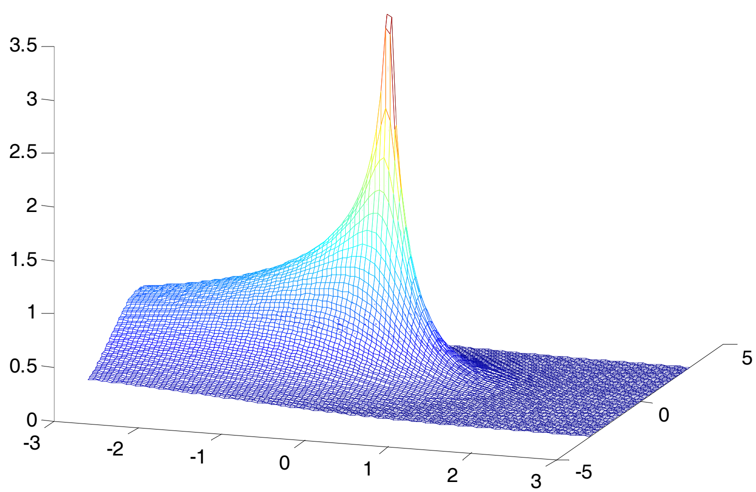
High diffusion



High reaction



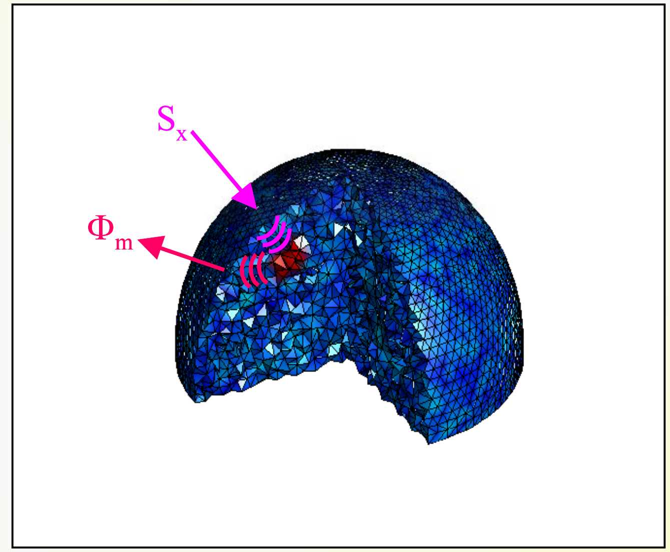
High convection



GOVERNING EQUATIONS

matrix formulation of coupled complex equations

$$
\begin{cases}
-\underline{\underline{\nabla}}^{t}\left(\underline{\underline{\mathbf{d}}}\,\underline{\underline{\nabla}}\,\underline{\mathbf{\Phi}}\right)+\underline{\underline{\mathbf{k}}}\,\underline{\mathbf{\Phi}}=\underline{\mathbf{S}} \quad \text{on} \quad \Omega \\[2ex]
\underline{\underline{\mathbf{D}}}\dfrac{\partial \underline{\mathbf{\Phi}}}{\partial n}+\underline{\underline{\mathbf{r}}}\,\underline{\mathbf{\Phi}}=\underline{0} \quad \text{on} \quad \partial\Omega
\end{cases}
$$



$$
\underline{\underline{\nabla}}\equiv\begin{bmatrix}\nabla & \underline{0}\\ \underline{0} & \nabla\end{bmatrix};\quad
\underline{\underline{\mathbf{d}}}\equiv\begin{bmatrix}D_x\underline{\underline{\mathbf{I}}} & \underline{\underline{\mathbf{0}}}\\ \underline{\underline{\mathbf{0}}} & D_m\underline{\underline{\mathbf{I}}}\end{bmatrix};\quad
\underline{\underline{\mathbf{D}}}\equiv\begin{bmatrix}D_x & 0\\ 0 & D_m\end{bmatrix};\quad
\underline{\underline{\mathbf{k}}}\equiv\begin{bmatrix}k_x & 0\\ -\beta & k_m\end{bmatrix};\quad
\underline{\underline{\mathbf{r}}}\equiv\begin{bmatrix}r_x & 0\\ 0 & r_m\end{bmatrix};
$$

$$
\underline{\mathbf{\Phi}}\equiv\begin{bmatrix}\Phi_x\\ \Phi_m\end{bmatrix};\quad
\underline{\mathbf{S}}\equiv\begin{bmatrix}S_x\\ 0\end{bmatrix}.
$$

BOUNDARY ELEMENT METHOD
FOR DIFFUSION-REACTION SYSTEMS

Multiply by an arbitrary matrix $\underline{\underline{\Psi}}^t$

$$\int_\Omega \underline{\underline{\Psi}}^t \left(-\underline{\underline{\nabla}}^t \left(\underline{\underline{\nabla}}\, \underline{\Phi} \right) + \underline{\underline{d}}^{-1} \underline{\underline{k}}\, \underline{\Phi} \right) d\Omega = \int_\Omega \underline{\underline{\Psi}}^t \underline{\underline{d}}^{-1} \underline{S}\, d\Omega$$

Integration by parts twice

$$\int_\Omega \left(-\underline{\underline{\nabla}}^t \left(\underline{\underline{\nabla}}\, \underline{\underline{\Psi}} \right) + \left(\underline{\underline{d}}^{-1} \underline{\underline{k}} \right)^t \underline{\underline{\Psi}} \right)^t \underline{\Phi}\, d\Omega + \int_{\partial\Omega} \left(-\underline{\underline{\Psi}}^t \frac{\partial \underline{\Phi}}{\partial n} + \frac{\partial \underline{\underline{\Psi}}}{\partial n} \underline{\Phi} \right) dS = \int_\Omega \underline{\underline{\Psi}}^t \underline{\underline{d}}^{-1} \underline{S}\, d\Omega$$

Term involving volume integral of the unknown $\underline{\Phi}$

"Green matrix"

$$-\underline{\underline{\nabla}}^t \left(\underline{\underline{\nabla}}\, \underline{\underline{\Psi}} \right) + \left(\underline{\underline{d}}^{-1} \underline{\underline{k}} \right)^t \underline{\underline{\Psi}} + \underline{\underline{\delta}} = 0$$

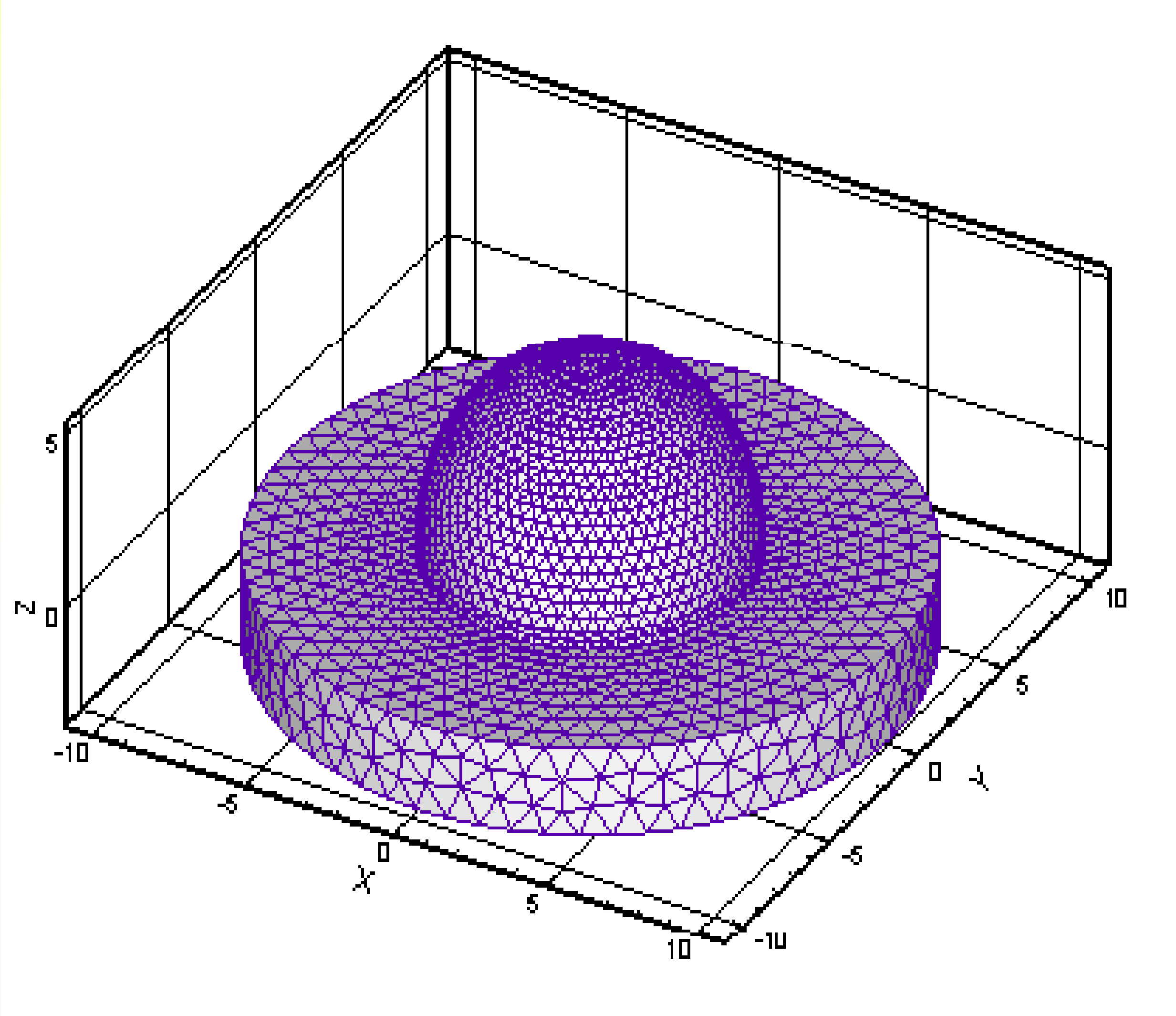
THE GREEN MATRIX

Choose $\underline{\underline{\Psi}}$ to be the ***"Green matrix"*** by putting a Dirac source $\underline{x}_0$ at the boundary points

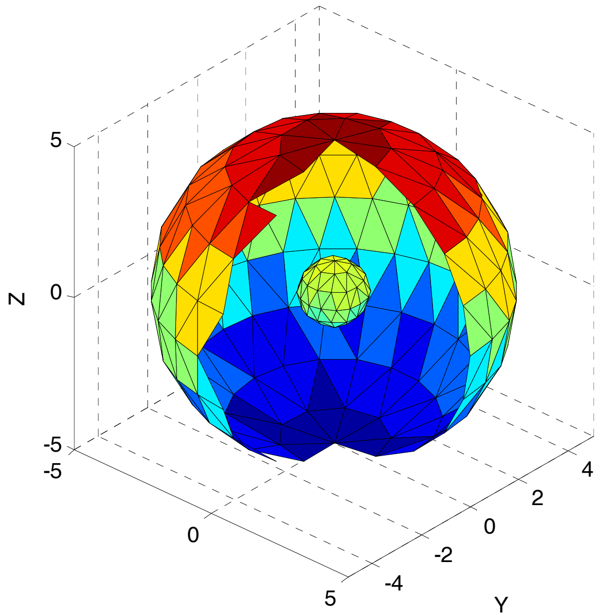
$$\underline{\boldsymbol{\Phi}}(\underline{\boldsymbol{x}}_0) + \int_{\partial\Omega} \left(-\underline{\underline{\boldsymbol{\Psi}}}^t \frac{\partial \underline{\boldsymbol{\Phi}}}{\partial \boldsymbol{n}} + \frac{\partial \underline{\underline{\boldsymbol{\Psi}}}}{\partial \boldsymbol{n}} \underline{\boldsymbol{\Phi}} \right) d\boldsymbol{S} = \int_{\Omega} \underline{\underline{\boldsymbol{\Psi}}}^t \underline{\underline{\mathbf{d}}}^{-1} \underline{\boldsymbol{S}}\, d\Omega$$

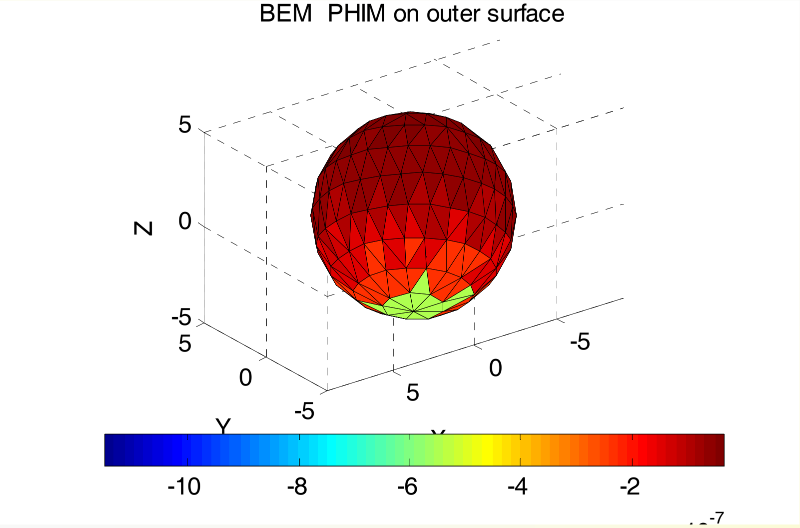
$$\underline{\underline{\boldsymbol{\Psi}}}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}_0) = \begin{bmatrix} \dfrac{\exp(-\boldsymbol{i}\lambda_1 |\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}_0|)}{4\pi |\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}_0|} & 0 \\ \alpha \left(\dfrac{\exp(-\boldsymbol{i}\lambda_2 |\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}_0|)}{4\pi |\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}_0|} - \dfrac{\exp(-\boldsymbol{i}\lambda_1 |\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}_0|)}{4\pi |\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}_0|} \right) & \dfrac{\exp(-\boldsymbol{i}\lambda_2 |\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}_0|)}{4\pi |\underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}_0|} \end{bmatrix}$$

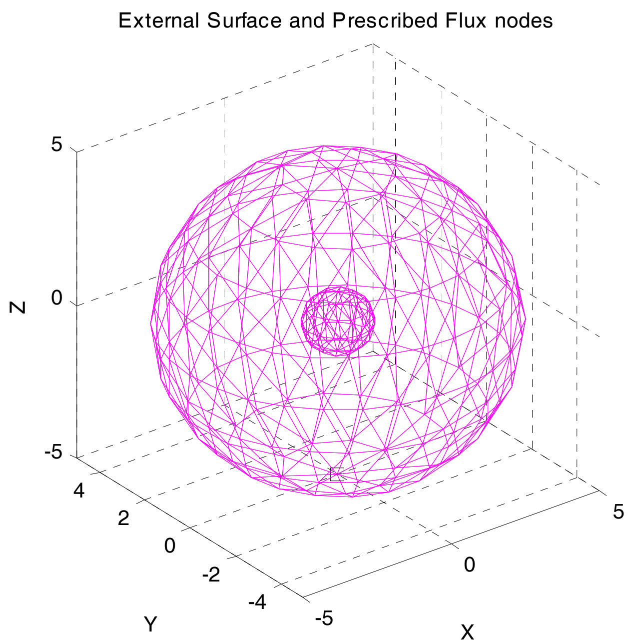
BOUNDARY ELEMENT MESH

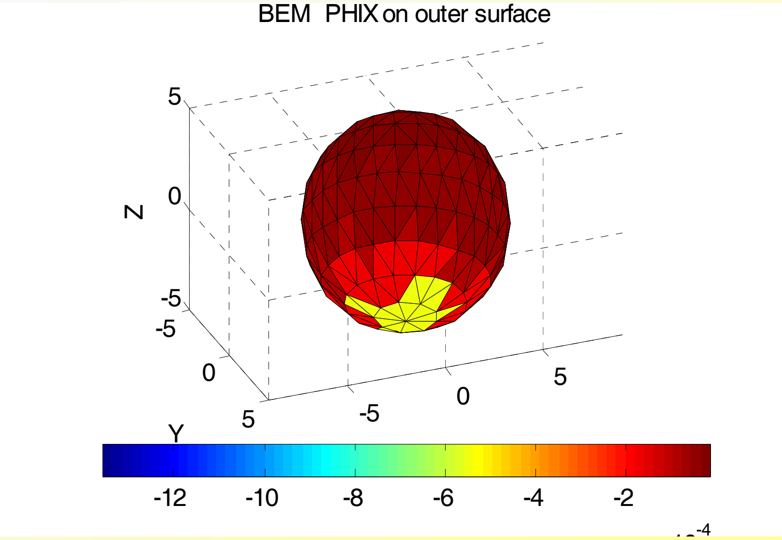


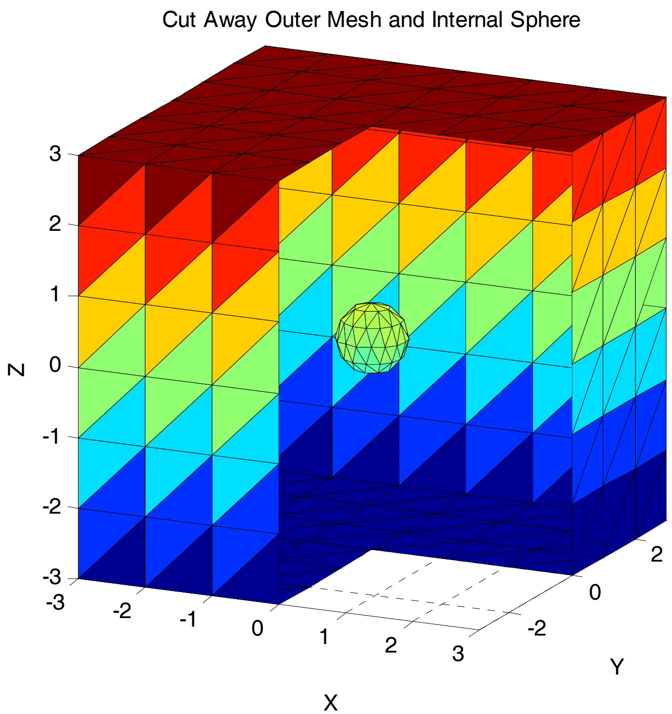
SOME APPLICATIONS

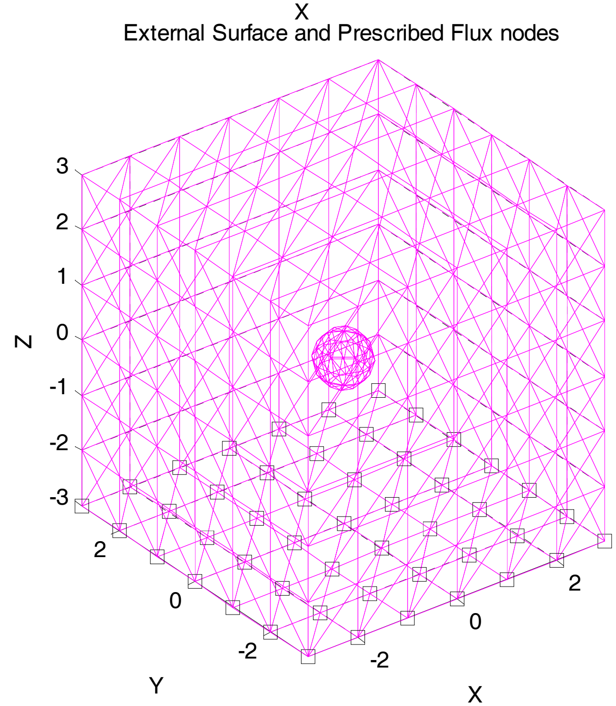


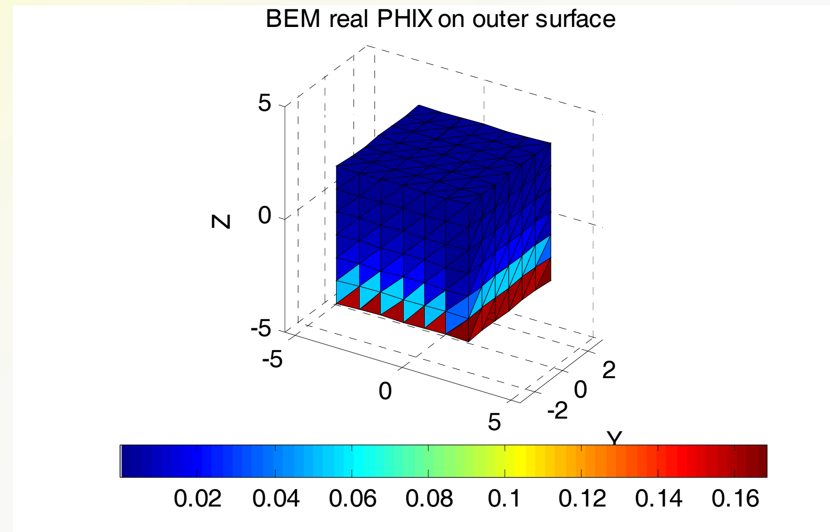


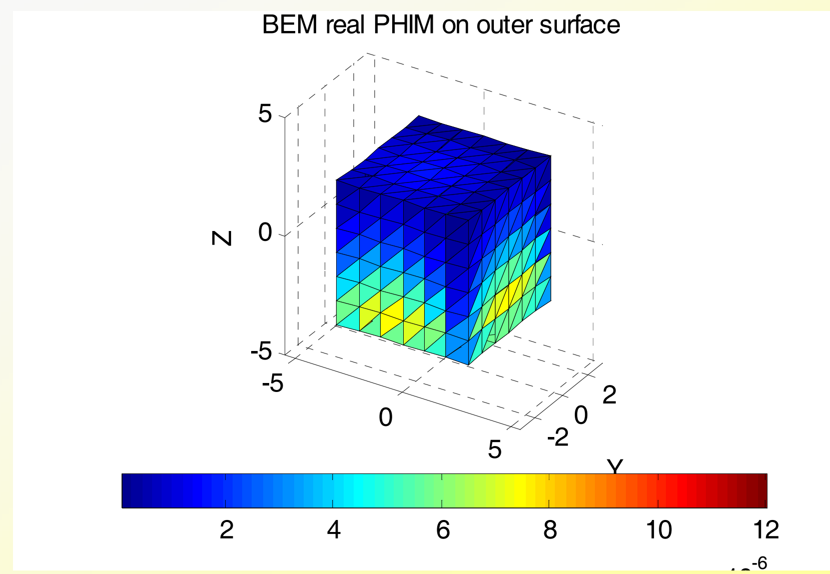





Cut Away Outer Mesh and Internal Sphere
Z
X
Y


External Surface and Prescribed Flux nodes
Z
Y
X


BEM real PHIX on outer surface
Z
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16


BEM real PHIM on outer surface
Z
2
4
6
8
10
12

ANY QUESTIONS ?