A SINGLE-DEGREE-OF -FREEDOM COLLOCATION SOLUTION TO THE TRANSPORT EQUATION

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SINGLE-DEGREE-OF-FREEDOM COLLOCATION METHOD (LHM)

The multi-dimensional transport equation is notoriously difficult to solve. As the value of the Peclet number becomes large, either dispersion of the higher harmonic modes is observed or excessive damping.

A combination of Lagrangian and cubic Hermite polynomial approximations is employed to derive a single-degree-of-freedom collocation method ***(LHM)***

EFFICIENCY OF **LHM**

- reduction in the degrees of freedom at each node when compared to Hermite-based collocation methods.

- decreasing of dispersion effects and damping when compared to classical Eulero-based methods (FEM /FDM) .

1D ADVECTIVE-DIFFUSIVE EQUATION

$$\begin{cases} \mathbf{L}(u) = 0 \\ u(x,t=0) = u_0(x) \\ u(0,t) = 0 \quad u(l,t) = 0 \end{cases} \qquad u_0(x) \in L_2[0,l]$$

with $$\mathbf{L} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} - D\frac{\partial^2}{\partial x^2}$$ Advective-diffusive operator

- $u(x,t)$ concentration at location x at the time t
- c velocity field
- D diffusion coefficient

SEMI-DISCRETE ADVECTIVE-DIFFUSIVE OPERATOR

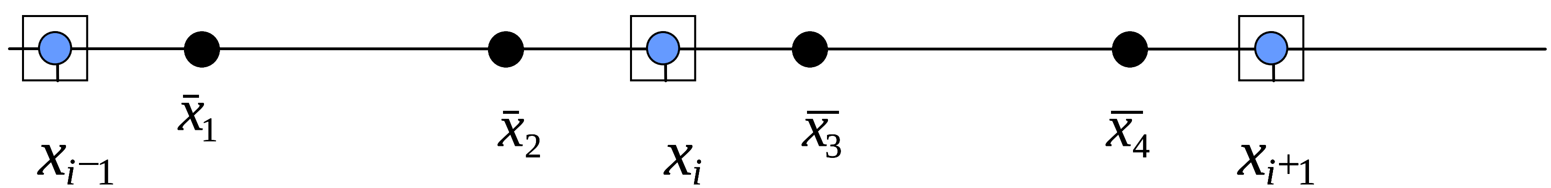
(spatial discretization)

Uniform mesh grid $\Omega_x = \{x_i, 1 \le i \le N_x\}$ $x_i = i\Delta x$

Surrounding mesh grid at location x_{i}

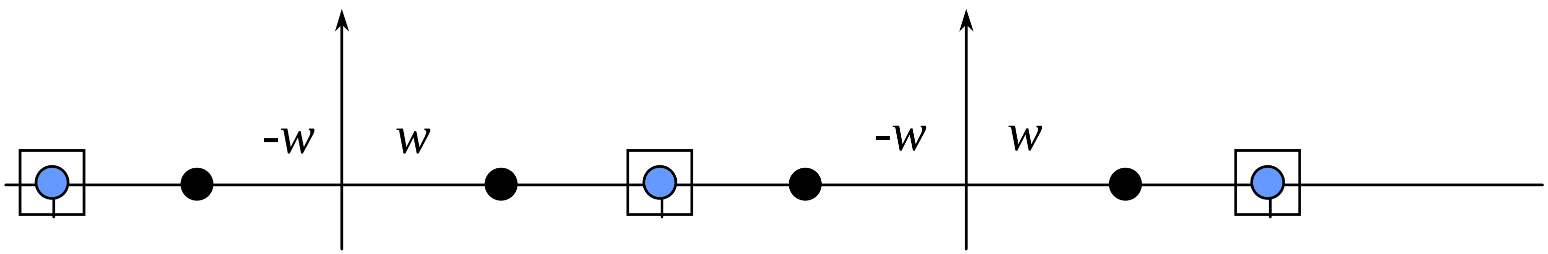
Collocation points

nodes



STEP 1 – ***HERMITE INTERPOLATION***

$$u(x,t)=\begin{cases} \boldsymbol{U}^L = \boldsymbol{H}_{0,i-1}(x)u_{i-1} + \boldsymbol{H}_{1,i}(x)\,u_i + \dfrac{\Delta x}{2}\overline{\boldsymbol{H}_{0,i-1}}(x)\left(\dfrac{\partial u}{\partial x}\right)_{i-1} + \dfrac{\Delta x}{2}\overline{\boldsymbol{H}_{1,i}}(x)\left(\dfrac{\partial u}{\partial x}\right)_{i} & x\in[x_{i-1},x_i] \\[2ex] \boldsymbol{U}^R = \boldsymbol{H}_{0,i}(x)u_{i} + \boldsymbol{H}_{1,i+1}(x)\,u_{i+1} + \dfrac{\Delta x}{2}\overline{\boldsymbol{H}_{0,i}}(x)\left(\dfrac{\partial u}{\partial x}\right)_{i} + \dfrac{\Delta x}{2}\overline{\boldsymbol{H}_{1,i+1}}(x)\left(\dfrac{\partial u}{\partial x}\right)_{i+1} & x\in[x_{i},x_{i+1}] \end{cases}$$



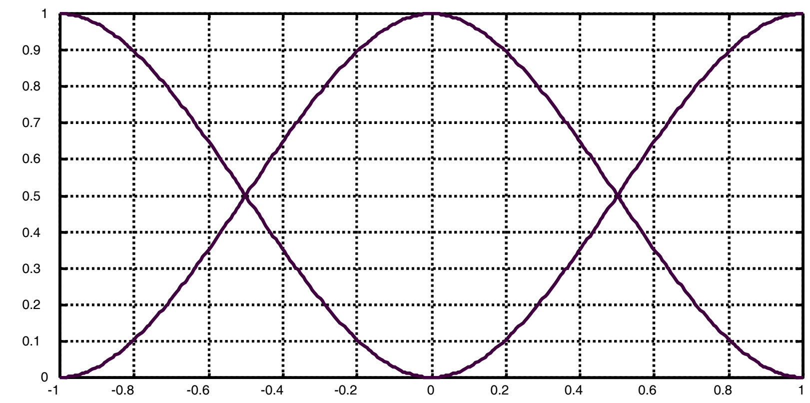
x_{i-1} x_i x_{i+1}

$u_{i-1},\left(\dfrac{\partial u}{\partial x}\right)_{i-1}$ $u_{i},\left(\dfrac{\partial u}{\partial x}\right)_{i}$ $u_{i+1},\left(\dfrac{\partial u}{\partial x}\right)_{i+1}$

unknown nodal values

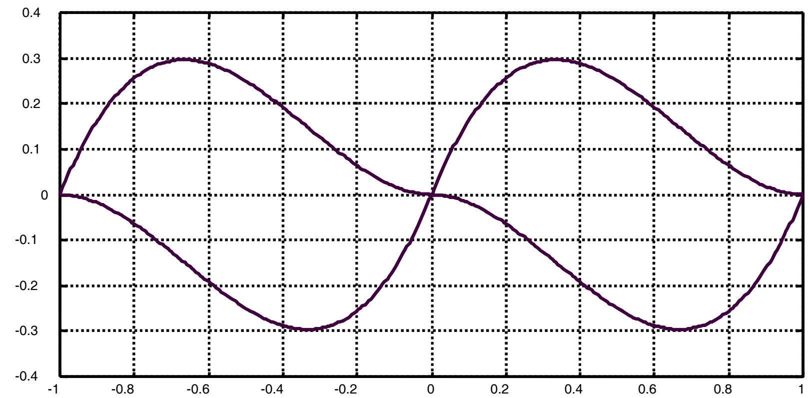
Hermite basis for

$$\{u_{n-1}, u_n, u_{n+1}\}$$



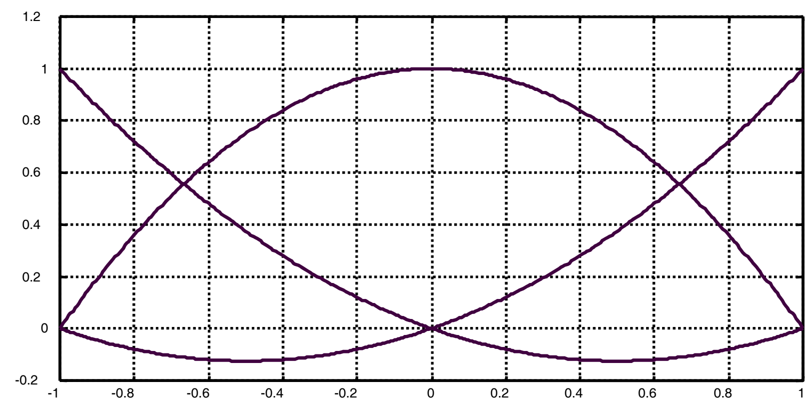
Hermite basis for

$$\left\{\left(\frac{\partial u}{\partial x}\right)_{n-1}, \left(\frac{\partial u}{\partial x}\right)_{n}, \left(\frac{\partial u}{\partial x}\right)_{n+1}\right\}$$



Lagrangian basis used for

the approximation of $\dfrac{\partial u}{\partial x}$



STEP 2- *LAGRANGIAN APPROXIMATION FOR THE SPATIAL DERIVATIVES*

$$\boldsymbol{U}(x,t)\Big|_{[x_{i-1},x_{i+1}]} = \boldsymbol{L}_{i-1}(x)u_{i-1} + \boldsymbol{L}_i(x)u_i + \boldsymbol{L}_{i+1}(x)u_{i+1}$$

$$\frac{\partial u}{\partial x} \cong \frac{\partial \boldsymbol{U}}{\partial x} = \frac{d\boldsymbol{L}_{i-1}}{d\,x}u_{i-1} + \frac{d\boldsymbol{L}_i}{d\,x}u_i + \frac{d\boldsymbol{L}_{i+1}}{d\,x}u_{i+1}$$

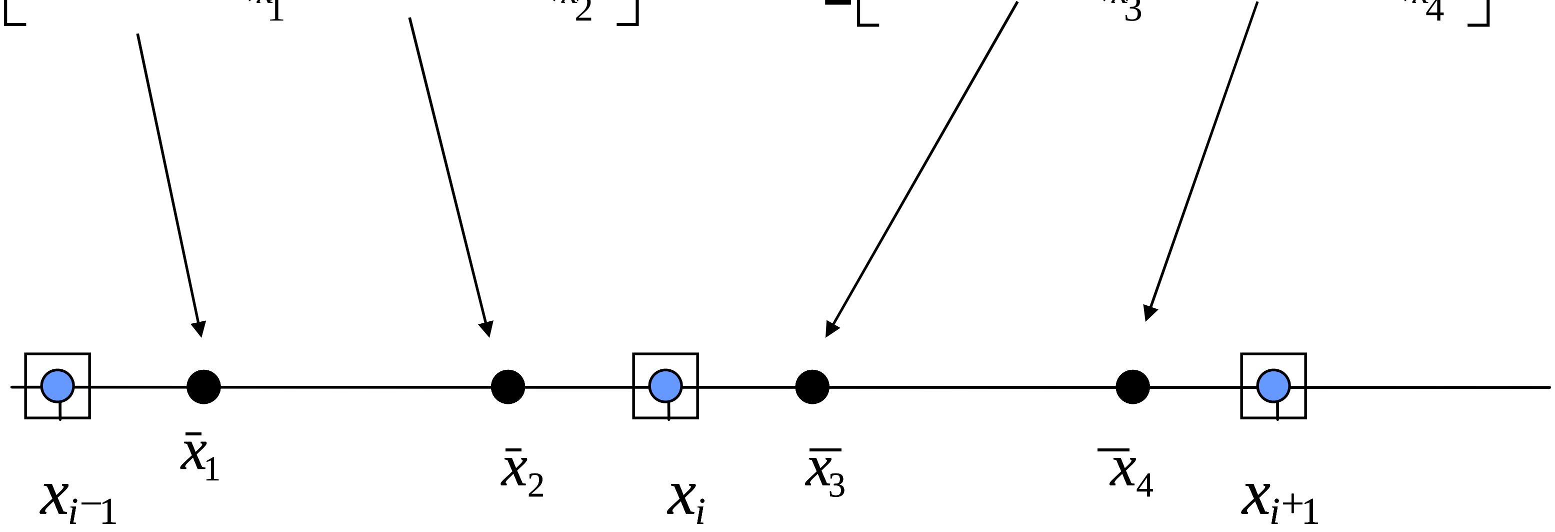
Approximation for the nodal-value spatial derivatives

$$\left(\frac{\partial u}{\partial x}\right)_{i-1} \cong \frac{-3u_{i-1} + 4u_i - u_{i+1}}{2\Delta x} \qquad \left(\frac{\partial u}{\partial x}\right)_i \cong \frac{u_{i+1} - u_{i-1}}{2\Delta x} \qquad \left(\frac{\partial u}{\partial x}\right)_{i+1} \cong \frac{u_{i-1} - 4u_i + 3u_{i+1}}{2\Delta x}$$

STEP 3 – ***WEIGHTED STRATEGY***

Defintion of the semi-discrete operator at the generic node

$$\hat{\mathbf{L}}_x(u)_i = \beta\frac{1}{2}\left[\mathbf{L}\left(\mathbf{U}^L\right)\Big|_{\bar{x}_1} + \mathbf{L}\left(\mathbf{U}^L\right)\Big|_{\bar{x}_2}\right] + (1-\beta)\frac{1}{2}\left[\mathbf{L}\left(\mathbf{U}^R\right)\Big|_{\bar{x}_3} + \mathbf{L}\left(\mathbf{U}^R\right)\Big|_{\bar{x}_4}\right]$$



β *up – wind parameter*

SEMI-DISCRETE SPATIAL APPROXIMATION

By imposing the vanishing of the discrete operator at each node the following ODE system is achieved

$$a_1 \frac{du_{i-1}}{dt} + a_2 \frac{du_i}{dt} + a_3 \frac{du_{i+1}}{dt} + b_1 u_{i-1} + b_2 u_i + b_3 u_{i+1} = 0 \qquad \forall i = 1,2,..,N_X$$

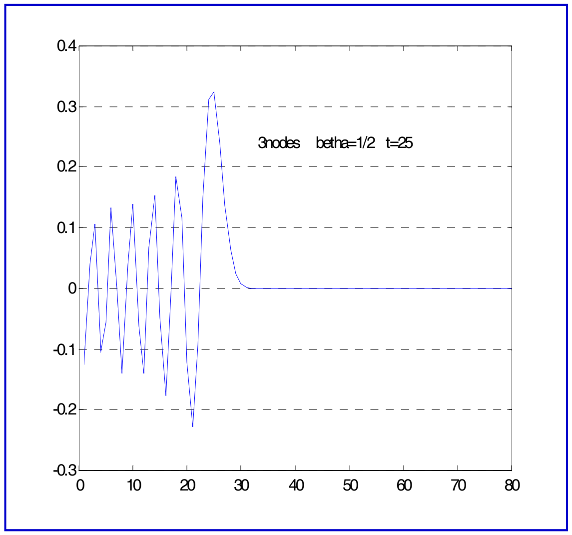
ANALYSIS OF THE FUNDAMENTAL SOLUTION

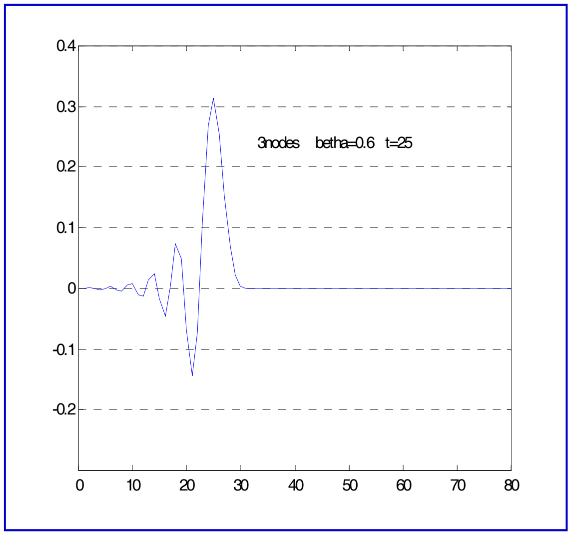
Analytical solution for initial condition $u_n(t=0) = \delta_{0,n} \quad \forall\, n \in Z$

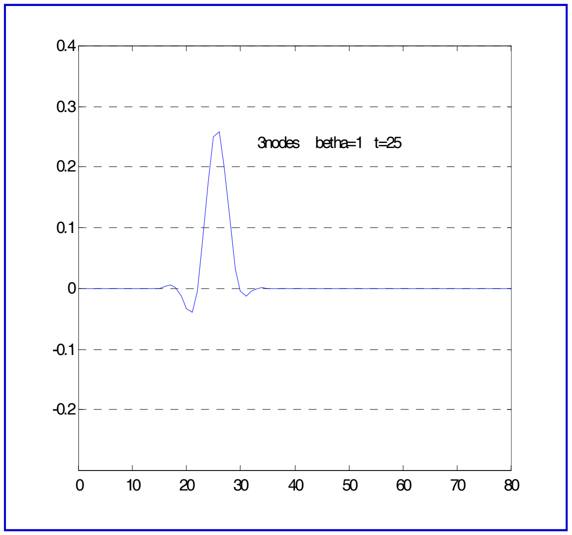
(Kronecher's symbol)

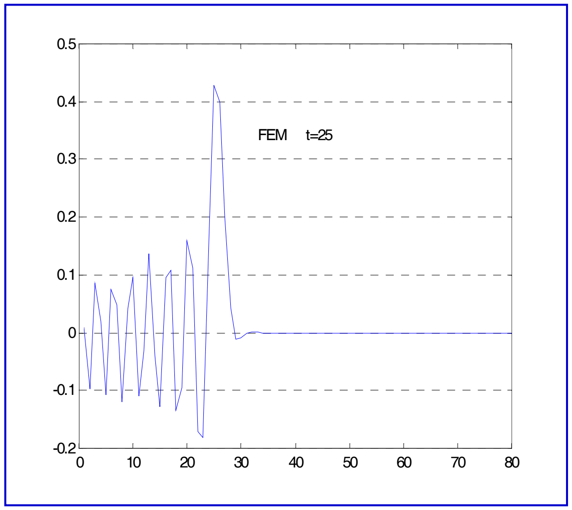
(by using Fourier Transform)

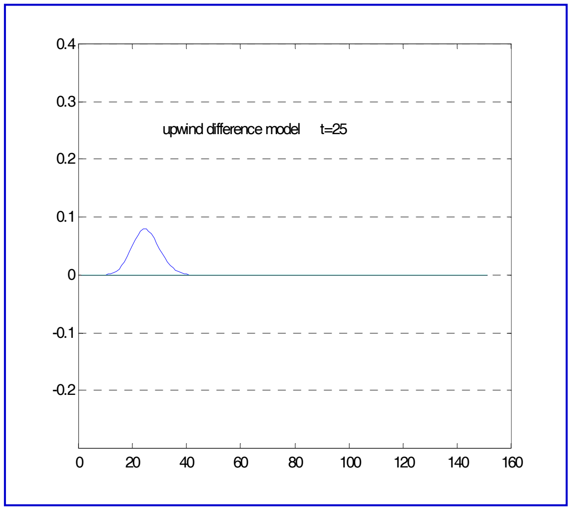
BEHAVIOR OF THE FUNDAMENTAL SOLUTION FOR LHM , FEM, FDM











FULLY DISCRETE ADVECTIVE-DIFFUSIVE OPERATOR

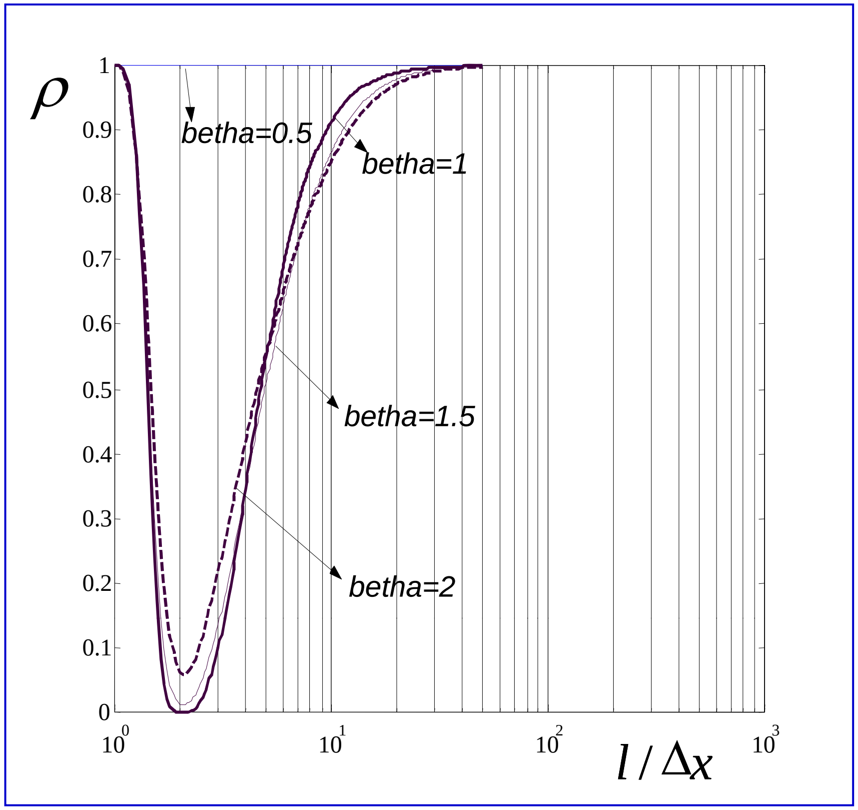
$$\hat{L}(u) = A_1 u_{i-1}^{n+1} + A_2 u_i^{n+1} + A_3 u_{i+1}^{n+1} + B_1 u_{i-1}^{n} + B_2 u_i^{n} + B_3 u_{i+1}^{n}$$

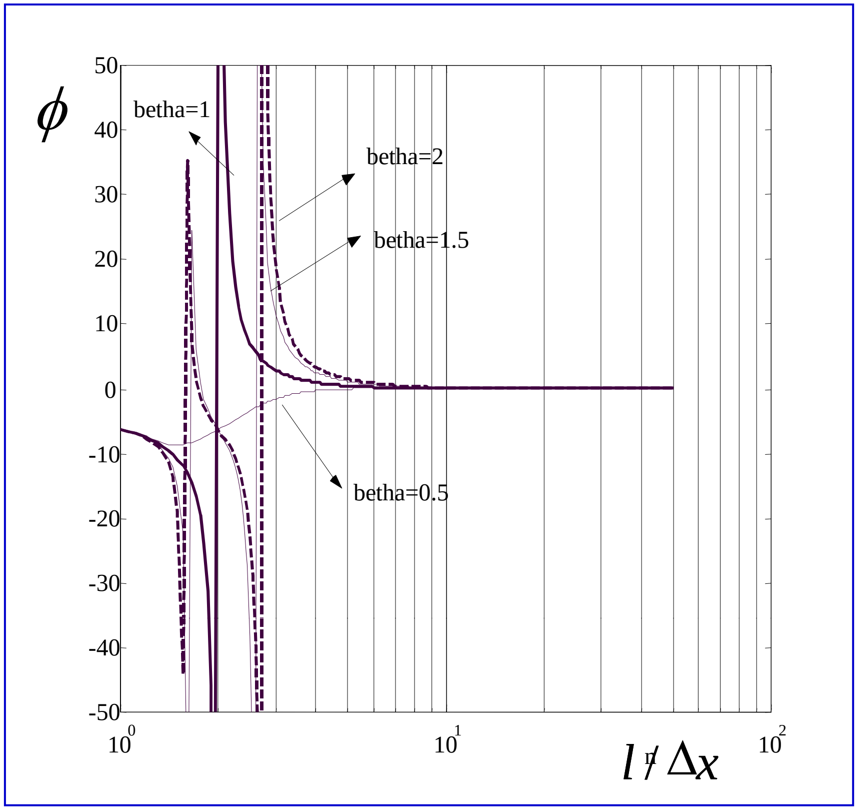
$$A_j = a_j / \Delta t + \gamma\, b_j\,, \quad B_j = -a_j / \Delta t + (1-\gamma)\, b_j\,, \qquad j = 1,2,3$$

Time-weight parameter γ $\quad [\gamma = 1/2 \quad \text{(Crank – Nicolson in time)}]$

Matrix-form $\quad \underline{\underline{A}}\,\underline{X}^{n+1} + \underline{\underline{B}}\,\underline{X}^{n} = \underline{C} \qquad \underline{\underline{A}}, \underline{\underline{B}}$ ***Tridiagonal***

STABILITY ANALYSIS AND SPECTRAL PROPERTIES





$$\hat{\mu} = \frac{\hat{u}_j^{n+N}}{\hat{u}_j^n}$$

numerical amplification factor

$$\mu = \frac{u_j^{n+N}}{u_j^n}$$

analytical amplification factor

$$\xi = \frac{\hat{\mu}}{\mu} = \rho\, e^{i\phi}$$

N - time steps to travel one wave length

CONVERGENCE ANALYSIS

$$\frac{c\,\Delta x}{D} < \infty \qquad\qquad \frac{c\,\Delta x}{D} = \infty$$

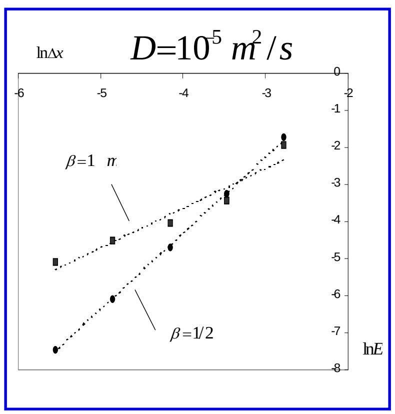
$$E(x,t) \approx \begin{cases} O(\Delta x^2 + \Delta t^2) & if \quad \beta = 1/2 \quad \forall\ w \in [0,1] \\ O(\Delta x + \Delta t^2) & if \quad \beta \neq 1/2 \quad \forall\ w \in [0,1] \end{cases}$$

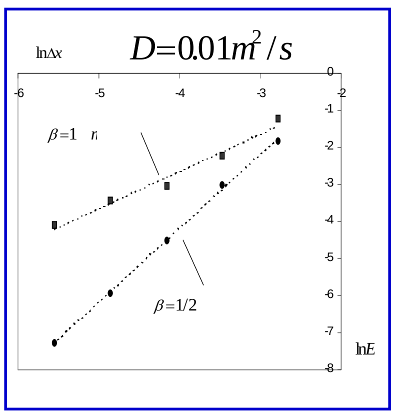
$$E(x,t) \approx \left[\begin{array}{lll} w^2 \neq 1/3 & O(\Delta x^2 + \Delta t^2) & \forall\ \beta \geq 1/2 \\ w^2 = 1/3 & \begin{cases} O(\Delta x^3 + \Delta t^2) & \beta > 1/2 \\ O(\Delta x^4 + \Delta t^2) & \beta = 1/2 \end{cases} & \end{array} \right.$$

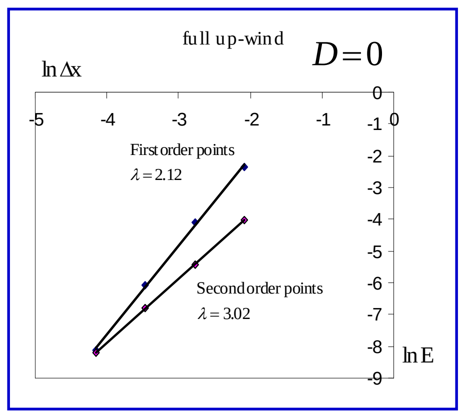
$\pm\, w$ Position collocation points

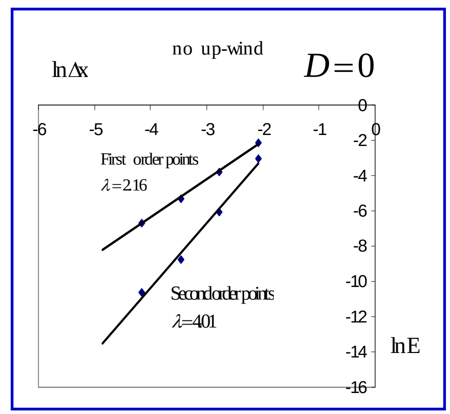
$\gamma = 1/2$ (Crank – Nicolson in time)

NUMERICAL TEST (Gaussian hill initial condition)



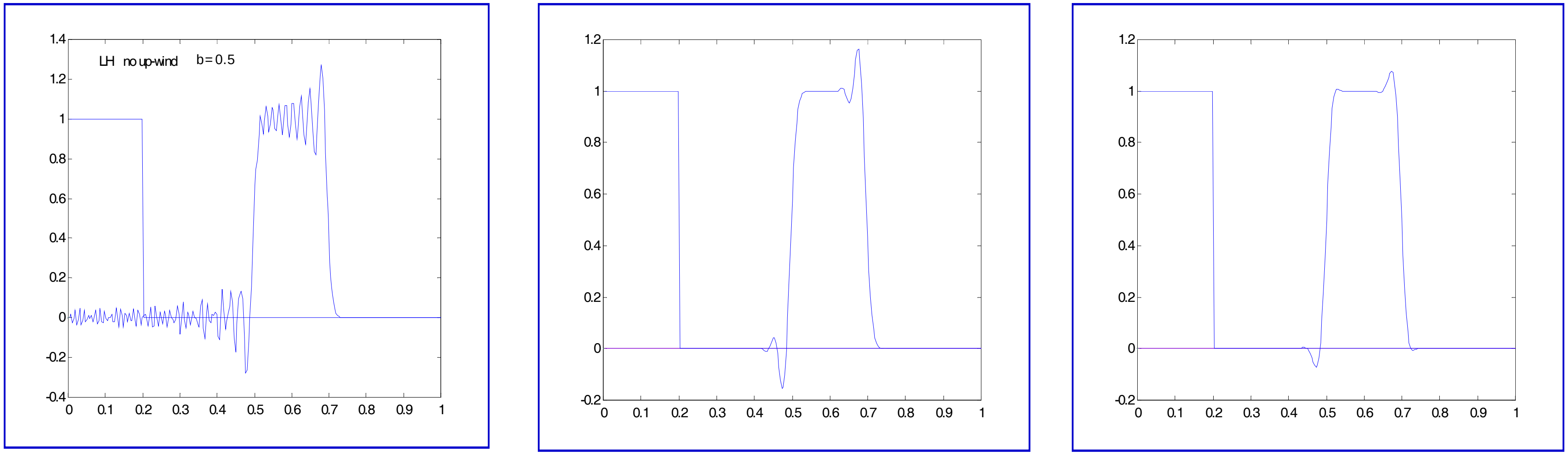




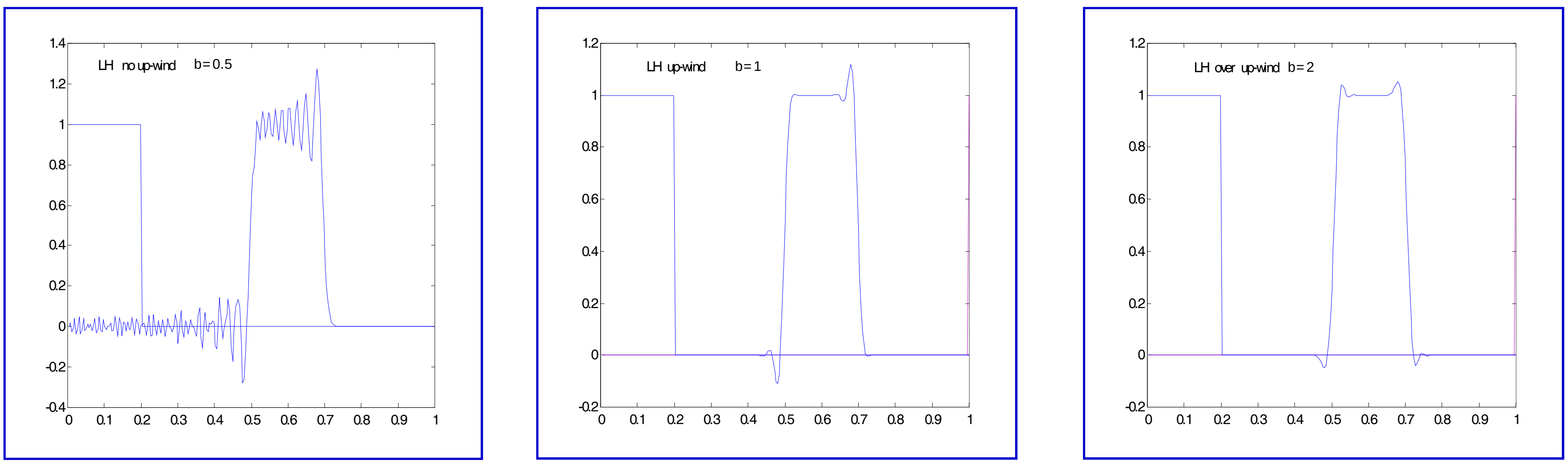


Super convergence for pure advection

1D SIMULATIONS

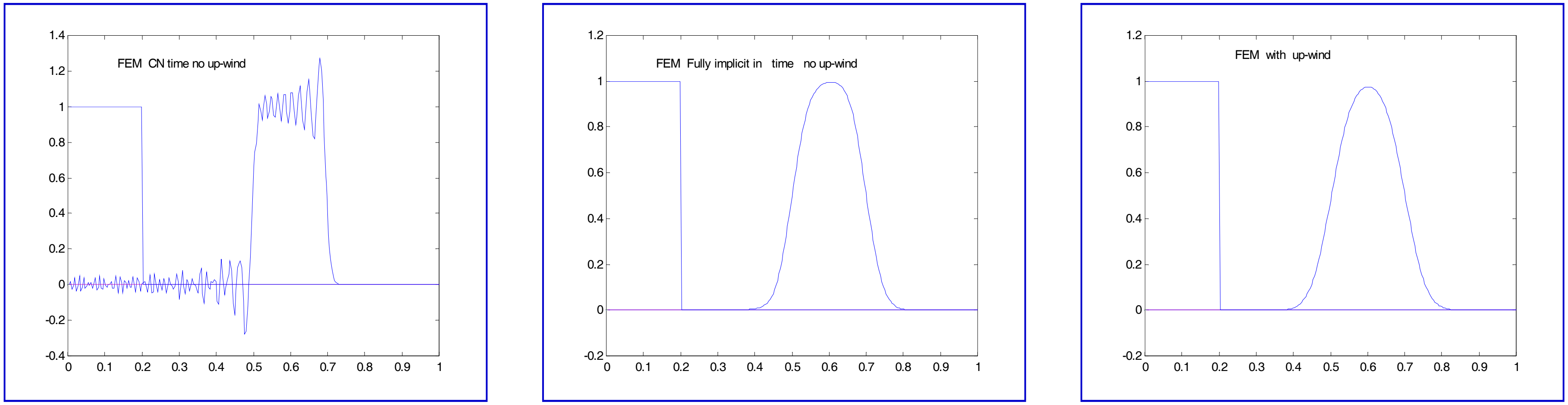


First order points collocation (one gaussian point in each interval) c=0.5 m dx=2^{-8} m dt=1/200 s

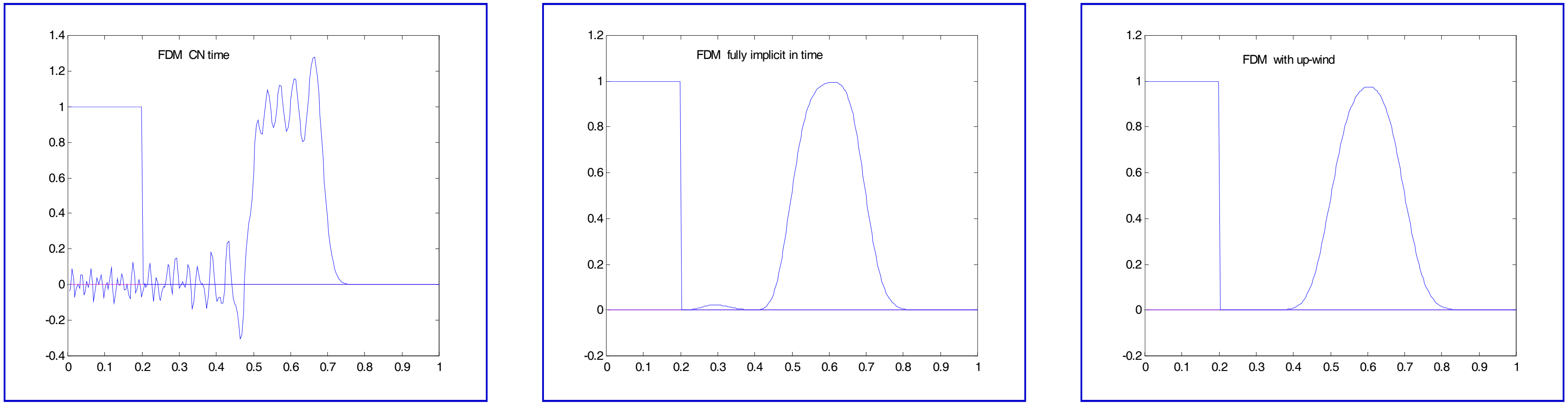


Second order points collocation (two gaussian points in each interval) c=0.5 m dx=2^{-8} m dt=1/200 s

1D SIMULATIONS - FEM/FDM

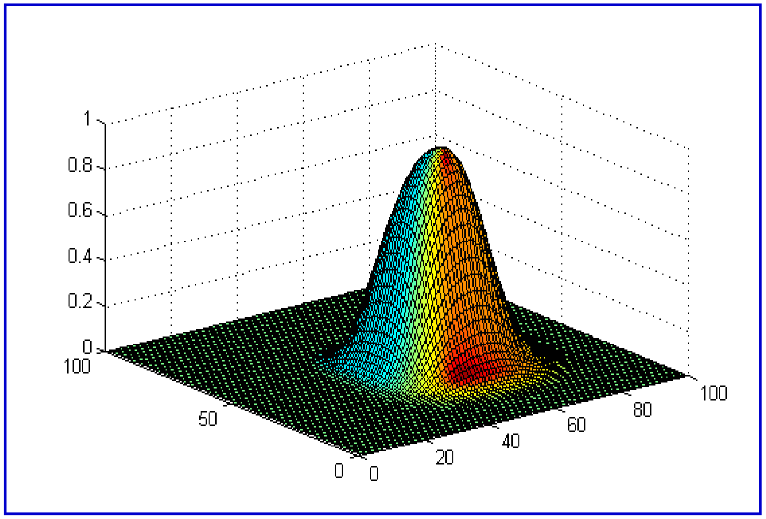


FEM from the left side CN in time, full implicit in time, with up-winding

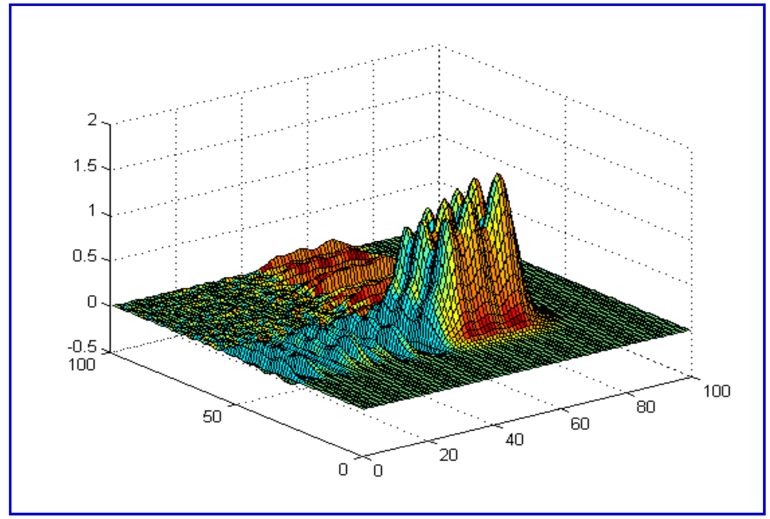


FDM from the left side CN in time, full implicit in time, with up-winding

2D SIMULATIONS



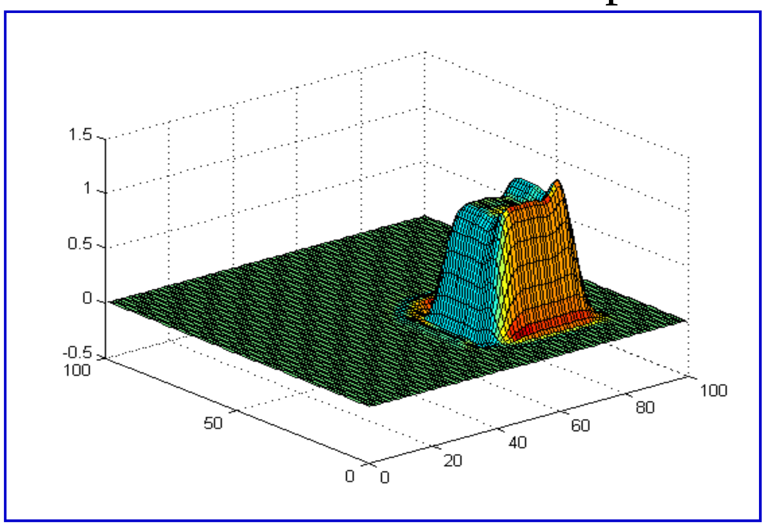
FDM up-wind



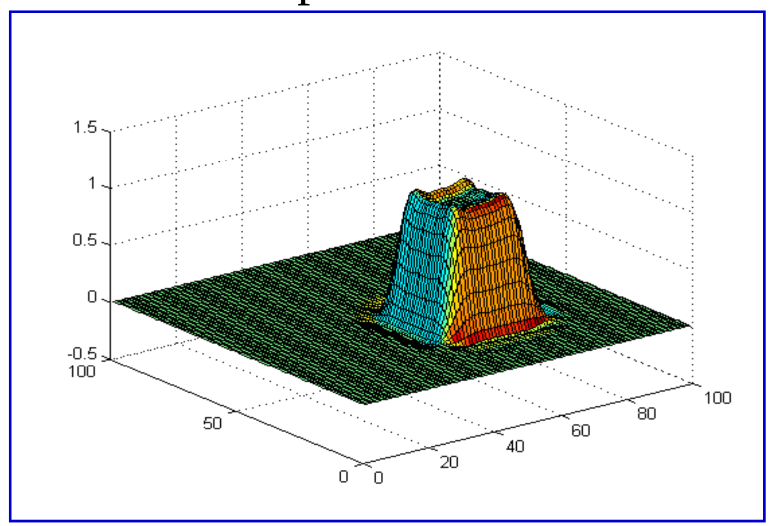
FDM no up-wind

TRANSLATING TOWER

LHM full up-wind



LHM over up-wind



CONCLUSIONS

- A single-degree-of-freedom Collocation method (LHM) is proposed based on a combination of Lagrangian and Hermite polynomials

-LHM reduces the number of nodal unknowns from 2 to 1 in 1D, from 4 to 1 in 2D when compared to Cubic Hermite Collocation

- spectral analysis shows that LHM has a very narrow-band spectrum: it damps only the spurious modes, preserving the behavior of the physical harmonic modes

- super convergence is found for the case of pure advection