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On wave groups in a Gaussian sea

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Abstract

In the context of Gaussian waves, if two successive wave crests of amplitude h_1 and h_2, respectively, are recorded in time at a fixed point $\mathbf{x}_0$ then in the limit of $h_1 \to \infty$ and $h_2 \to \infty$, with probability approaching 1, a wave group has passed closed by the point $\mathbf{x}_0$ at the apex of its development stage, giving rise to an isolated extreme crest. The two large successive wave crests occur at $\mathbf{x}_0$ during the initial phase of decay of the wave group and they are lagged in time by $T_2^* + O(h_1^{-1}, h_2^{-1})$, T_2^* being the abscissa of the second absolute maximum of the time covariance function $\psi(T)$ of the surface displacement.

Thus, either an isolated extreme crest event or two consecutive extreme crest events are particular realizations of the space–time evolution of a wave group, in agreement with the theory of quasi determinism of Boccotti [2000. Wave Mechanics for Ocean Engineering. Elsevier, Oxford].

This result is of relevant interest for offshore engineering. Firstly, the design of offshore structures resisting to a double wave impact can be based on the wave forces generated by the mechanics of a single wave group. On the other hand, in the context of nonlinear water waves, extreme events and their probability of occurrence can be investigated by studying the nonlinear evolution of a wave group.

Keywords: Successive wave crests; Gaussian sea; Quasi-determinism; Wave group; Conditional probability; Extreme event; Nonlinear evolution

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1. Introduction

In the context of Gaussian waves, in the early seventies Lindgren proved that in the time domain, locally to a very high crest, the surface displacement tends to assume the shape of the autocovariance function $\psi(T) = \langle \eta(t)\eta(t+T) \rangle$ where $\langle \cdot \rangle$ is the time average operator (cf. Lindgren, 1970, 1972). Tromans et al. (1991) used this time-domain formulation to analyze wave measurements and renamed it as 'new wave theory'. Wave statistics has been studied by Longuet-Higgins (1952) who proved that the wave heights of a narrow-band Gaussian sea are distributed according to the Rayleigh form. Because of the symmetry of the Gaussian sea state both crest and trough distributions follow the same Rayleigh law for narrow-band spectra. For more general Gaussian processes with finite-band spectra, it is well known that the Rayleigh distribution is an upper bound for the probability of exceedance of both crest heights and crest-to-trough wave heights. It is also well known that the Rayleigh law tends to be asymptotically exact in the limit of large crest amplitudes (cf. Sun, 1993; Maes and Breitung, 1997). As regard to the crest-to-trough wave heights, variants of the Rayleigh distribution which take into account the effects due to the finite bandwidth of the spectrum have been proposed by Longuet-Higgins (1980) and Naess (1985). A rigorous derivation of the exact asymptotic expression for the probability of exceedance of crest-to-trough wave heights, irrespective of bandwidth of the spectrum, have been derived for the first time, in the eighties, by Boccotti (1981, 1982, 1983, 1989) as a corollary of his theory of quasi-determinism. Boccotti (1997, 2000) formulated his theory revealing the mechanics of three-dimensional wave groups and their relation to the occurrence of extreme waves in a Gaussian sea. The theory was verified in the nineties with some small-scale field experiments both for waves in an undisturbed field (cf. Boccotti et al., 1993a) and for waves interacting with structures (cf. Boccotti et al., 1993b). An alternative approach for the derivation of the quasi-determinism theory was proposed by Phillips et al. (1993a) who also obtained a field verification off the US Atlantic coast (Phillips et al., 1993b). There are two versions of the theory of quasi-determinism: the first version deals with the extreme crest height, whereas the second one deals with the extreme wave height. Both the versions are congruent to each other because they both reveal that either an extreme crest height or a wave height are particular realizations of the evolution of a well defined wave group (cf. Boccotti, 2000, 482pp). In particular, an extreme crest occurs at the point $\mathbf{x}_0$ when a wave group passes through $\mathbf{x}_0$ with the crest of its central wave exactly at the envelope center. The wave group has reached its maximal contraction at the point $\mathbf{x}_0$ and after it tends to decay. If an extreme wave height is recorded at the point $\mathbf{x}_0$ instead, it means that the wave group has reached its maximal contraction before the point $\mathbf{x}_0$. In this case, the wave group passes through the point $\mathbf{x}_0$ in its initial phase of decay and the zero downcrossing of the central wave coincides with the envelope center.

As corollary of his theory, Boccotti (1989, 2000) derived the asymptotic form of the probability distribution of the crest-to-trough wave height H_{w} as

$$\Pr(H_{\mathrm{w}} > H) = c \exp\left(-\frac{H^2}{4\sigma^2(1+\psi^*)}\right), \quad \frac{H}{\sigma} \to \infty,$$

where the factor c is given by

$$c = \frac{1 + \ddot{\psi}^*}{\sqrt{2\ddot{\psi}^*(1 + \psi^*)}},$$

where

$$\psi^* = \frac{|\psi(T^*)|}{\psi(0)}, \quad \ddot{\psi}^* = \frac{\ddot{\psi}(T^*)}{|\ddot{\psi}(0)|}.$$

Here, dots denote time derivatives, T^* is the abscissa of the first absolute minimum of the autocovariance function $\psi(T)$ and ψ^* is a narrow bandedness parameter. This probability of exceedance is asymptotically exact in the limit of large wave height amplitudes H, i.e. $H/\sigma \to \infty$, irrespective of the bandwidth of the spectrum. In the limit of narrow-band spectra, that is $\psi^* \to 1$, the factor $c \to 1$ and Boccotti distribution simplifies to the distribution proposed by Naess (1985) where he assumed $T^* \simeq T_{\mathrm{m}}/2$, T_{m} being the mean wave period. Naess distribution is asymptotically correct in the narrow-band limit since $T^* \to T_{\mathrm{m}}/2$ reducing down to the well known Rayleigh form. The exact narrow-band limit form of the crest-to-trough height distribution has been obtained by Tayfun (1981, 1990). Boccotti distribution agrees very well with Monte Carlo simulations and experimental data and it is approximately exact for $H/\sigma > 3.5$ (cf. Boccotti, 2000, 308pp).

In this paper the theory of quasi-determinism of Boccotti is revisited in order to emphasize a special aspect of it: the mechanics of the single wave group. The main goal is to show that the single wave group can be thought as a '*gene*' of a Gaussian sea when the interest is in the dynamics of the surface displacement at high amplitude levels. Thus extreme events are most likely to occur because of the dynamics of a single wave group.

In the context of Boccotti's theory, Fedele (2005) showed that if two large successive wave crests of amplitude h_1 and h_2, respectively, are recorded in time at a fixed point $\mathbf{x}_0$ then, in the limit of $h_1 \to \infty$ and $h_2 \to \infty$, with probability approaching 1, the two successive wave crests are lagged in time by $T_2^* + O(h_1^{-1}, h_2^{-1})$, T_2^* being the abscissa of the second absolute maximum of the autocovariance function $\psi(T)$ of the surface displacement. As a corollary, Fedele showed that the joint probability density function of two successive wave crests follows asymptotically a Rayleigh distribution.

In this paper we first revisit the time domain analysis of Fedele (2005) and then some results from Monte Carlo simulations are presented to validate his theoretical conclusions. The analysis is then extended to the space–time domain and it will surprisingly reveal a relation between the occurrence of two successive wave crests and the evolution of a wave group.

If two extreme consecutive crests occur at a certain location $\mathbf{x}_0$ in time, what happened in the space–time neighborhood of $\mathbf{x}_0$? What caused the formation of two extreme crests at $\mathbf{x}_0$? An infinite set of feasible wave scenarios can be proposed that may cause the occurrence of two consecutive wave crests. For an example, it may be possible that two, or more than two wave groups, travelling along different directions with different speeds, meet at $\mathbf{x}_0$ and two consecutive wave crest are formed in time at $\mathbf{x}_0$. It may also be possible that there are two wave groups which are following each other at different speeds so that the fast group will catch up with the slow group when two wave crests occur at $\mathbf{x}_0$. The theory of quasi-determinism of Boccotti excludes all the previous scenarios because they are unlikely to occur. The space–time domain analysis presented in this paper surprisingly reveals that if two consecutive exceptionally high wave crests are recorded at a point $\mathbf{x}_0$ during a sea storm, most probably, a well defined single wave group has transited at that point in the early stage of its decay. Thus, it is the same wave group that generates either an isolated extreme crest events or two consecutive extreme crest events; in the former case the wave group is in its configuration of max development, whereas in the latter case the wave group is beginning its phase of decay.

The relevance of this result is twofold. Firstly, it reveals that the dynamics of a Gaussian sea at high amplitude levels is governed by the evolution of a single wave group. On the other hand, this result is believed to be of relevant interest for offshore engineering. The design of offshore structures resisting to a double wave impact, can be based on the wave forces generated by the mechanics of a single wave group.

Note that the theory of quasi-determinism presented here has also been extended to investigate the weakly nonlinear evolution of a wave group due to both second-order effects (cf. Fedele and Arena, 2005; Arena and Fedele, 2005) and third-order effects as the four wave resonance interaction (cf. Fedele, 2004, 2006) providing new form of distributions for the wave crest statistics. Thus, the theory of quasi-determinism is a powerful mathematical means for studying extreme events and their probability of occurrence in the context of linear and nonlinear water waves.

2. The theory of quasi-determinism

In the following Boccotti's theory is presented for the general case of three-dimensional random waves (cf. Boccotti, 1989, 2000). Assume that a large wave crest of amplitude H have been recorded at point $\mathbf{x} = \mathbf{x}_0 = (x_0, y_0)$ at time $t = t_0$ and define σ as the standard deviation of the surface displacement. Boccotti has proven that as $H/\sigma \to \infty$, with probability approaching one, a well defined wave group has passed through the point $\mathbf{x} = \mathbf{x}_0$ when the apex of its development stage occurred at time $t = t_0$. If $H/\sigma \to \infty$, i.e. the crest is very high with respect to the mean crest height, then with probability approaching 1, the surface displacement in the neighborhood of $\mathbf{x} = \mathbf{x}_0$ and $t = t_0$ is asymptotically equal to the deterministic form

$$\xi_{\mathrm{det}}(\mathbf{x}_0 + \mathbf{X}, t_0 + T) = H \frac{\Psi(\mathbf{X}, T)}{\Psi(\mathbf{0}, 0)}, \tag{1}$$

Here, $\mathbf{X} = (X, Y)$ and $\Psi(\mathbf{X}, T)$ is the space–time covariance given by

$$\Psi(\mathbf{X}, T) = \langle \xi(\mathbf{x}_0, t)\xi(\mathbf{x}_0 + \mathbf{X}, t + T) \rangle, \tag{2}$$

where

$$\langle f(t)\rangle = \lim_{\mathcal{T}\to\infty}\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} f(t)\,\mathrm{d}t$$

is the time average. Note that $\Psi(\mathbf{0},0)=\sigma^2$. As an application consider unidirectional Gaussian waves along the X-axis. According to the theory of sea states (see Appendix), for a given wave spectrum $S(\omega)$, the space–time covariance function $\Psi(X,T)$ can be computed as

$$\Psi(X,T) = \int_0^\infty S(\omega)\cos(kX-\omega T)\,\mathrm{d}\omega, \tag{3}$$

where ω is the wave frequency related to the wave number k through the linear dispersion relation $\omega^2/g = k\tanh(kd)$ with g as the acceleration due to gravity. Then from Eq. (1) the unidirectional wave group is given by

$$\xi_{\mathrm{c}}(X,T) = H\int_0^\infty S(\omega)\cos(kX-\omega T)\,\mathrm{d}\omega. \tag{4}$$

This wave group evolves in the X-direction so that the highest crest of amplitude H occurs at $X=0$ and $T=0$. The wave group then begins its phase of decay and the highest wave occurs with a crest-to-trough amplitude $H_{\mathrm{w}} = H\sqrt{2(1-\psi^*)}$, T^* being the abscissa of the first absolute minimum of the time covariance function $\psi(T)=\Psi(0,T)$. Thus it is the same wave group in Eq. (4) that forms either the highest crest or the highest wave height. In the former case the wave group is in its configuration of max development, whereas in the latter case the wave group is in its configuration of initial phase of decay. Boccotti showed that in the limit of $H_{\mathrm{w}}/\sigma\to\infty$, the highest wave, with probability approaching 1, has a crest-to-trough period T_{w} almost equal to T^* and the crest and trough amplitudes tend to be equal. These analytical results are in perfect agreement with Monte Carlo simulations of a Gaussian sea with mean JONSWAP spectrum (cf. Hasselmann et al., 1973). In Fig. 1 the data points $(T_{\mathrm{w}}/T^*, H_{\mathrm{w}}/\sigma)$ obtained from Monte Carlo simulations are plotted. As one can see, for larger wave amplitudes H_{w}/σ the ratio T_{w}/T^* is closer to 1. From Fig. 2, the data points $(H_{\mathrm{cr}}/H_{\mathrm{tr}}, H_{\mathrm{w}}/\sigma)$ obtained from Monte Carlo simulations are plotted where H_{cr} and H_{tr} are the crest and trough amplitudes, respectively. For $H_{\mathrm{w}}/\sigma \gg 1$, the ratio $H_{\mathrm{cr}}/H_{\mathrm{tr}}$ is closer to 1 implying the crest–trough symmetry of the highest wave. The JONSWAP spectrum used in the Monte Carlo simulations is in the following form:

$$S(\omega) = Ag^2\omega_{\mathrm{p}}^{-5}\left(\frac{\omega}{\omega_{\mathrm{p}}}\right)^{-5}\exp\left[-\frac{5}{4}\left(\frac{\omega}{\omega_{\mathrm{p}}}\right)^{-4}\right]\exp\left\{\ln\gamma\exp\left[-\frac{(\omega-\omega_{\mathrm{p}})^2}{2\chi_2^2\omega_{\mathrm{p}}^2}\right]\right\}.$$

Here, ω_{p} is the peak frequency, A is the Phillips parameter, γ is the enhancement coefficient. For typical wind waves, one can assume $\gamma=3.3$ and $\chi_2=0.08$. For $\gamma=1$ and $A=0.0081$ the Pierson–Moskowitz spectrum is recovered. Hereafter $\omega_{\mathrm{p}}=2\pi/T_{\mathrm{p}}$, T_{p} being the peak period and L_{p} the correspondent wavelength. Note that the moments of the JONSWAP spectrum exist as far as m_3. By cutting off the

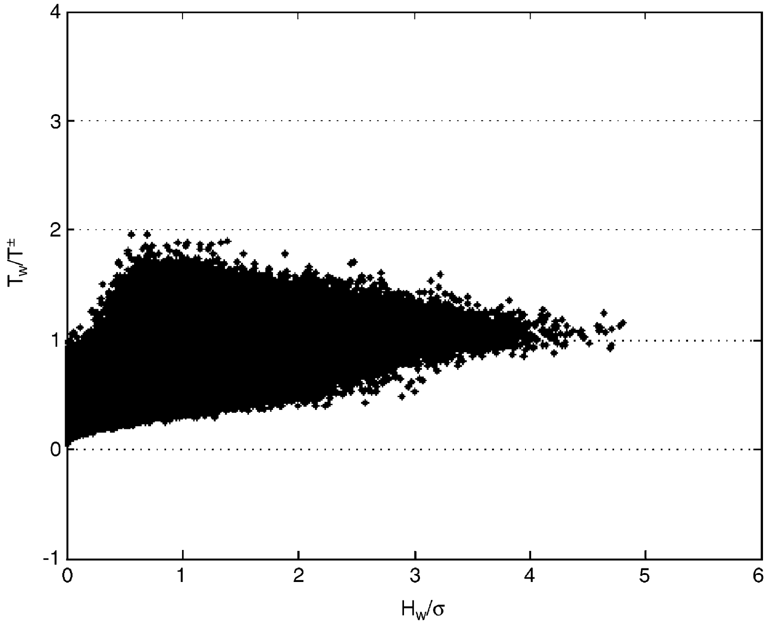

Fig. 1. Data points $(T_w/T^*, H_w/\sigma)$ obtained from Monte Carlo simulations of a Gaussian sea with mean JONSWAP spectrum. Here, T_w is the crest-to-trough period and H_w is the wave amplitude.

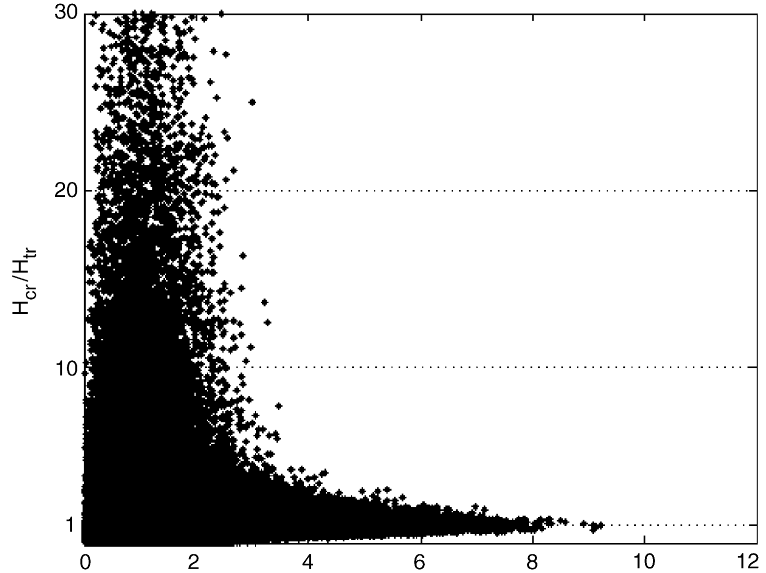

Fig. 2. Data points $(H_{cr}/H_{tr}, H_w/\sigma)$ obtained from Monte Carlo simulations of a Gaussian sea with mean JONSWAP spectrum. Here, H_{cr} and H_{tr} are the crest and trough amplitudes, respectively, and H_w is the wave amplitude.

high-frequency tails, one can define a new spectrum over a compact support and all the moments then exist. As pointed out by Boccotti (2000, 296pp) 'the high frequency term does not alter the crest elevation, nor the trough depth, nor the time interval between the crest and trough, nor the wave period. It simply ruffles the wave surface with a lot of very small ripples'. In the applications, the JONSWAP spectrum $S(\omega)$ can be considered in the frequency range $\omega \in [0, 6\omega_p]$ when the interest is in the analysis of the wave crest, wave trough or wave height.

3. The occurrence of two successive wave crests in the time domain

Let us consider the recorded time-series $\eta(t) = \xi(\mathbf{x}_0, t)$ of the surface displacement $\xi(\mathbf{x}, t)$ at any fixed point $\mathbf{x} = \mathbf{x}_0 = (x_0, y_0)$ in a Gaussian wave field. Set t_0 as an arbitrary time instant, h_1 and h_2 as wave crest heights and T_2^* as the abscissa of the second absolute minimum of the autocovariance $\psi(T) = \Psi(\mathbf{0}, T)$. Fedele (2005) showed that the conditions

$$\eta(t_0) = h_1 \quad \text{and} \quad \eta(t_0 + T_2^*) = h_2 \tag{5}$$

are necessary and sufficient for the occurrence of two successive wave crests in the limit of h_1/σ and h_2/σ approaching infinity. Conditions (5) are sufficient because the conditional p.d.f. of $\eta(t)$ at time $t_0 + T$

$$p[\eta(t_0 + T) = u/\eta(t_0) = h_1, \eta(t_0 + T_2^*) = h_2] \tag{6}$$

tends to a delta function $\delta[u - \eta_c(t_0 + T)]$ centered at the conditional mean $\eta_c(t_0 + T)$:

$$\eta_c(t_0 + T) = C_1\psi(T) + C_2\psi(T - T_2^*) \tag{7}$$

as both $h_1/\sigma \to \infty$ and $h_2/\sigma \to \infty$, where the coefficients C_1 and C_2 are given by

$$C_1 = \frac{h_1\psi(0) - h_2\psi(T_2^*)}{\psi^2(0) - \psi^2(T_2^*)}, \quad C_2 = \frac{h_2\psi(0) - h_1\psi(T_2^*)}{\psi^2(0) - \psi^2(T_2^*)}. \tag{8}$$

This implies that, in the limit of $h_1/\sigma \to \infty$ and $h_2/\sigma \to \infty$, all the realizations of the Gaussian sea satisfying conditions (5), with probability approaching one, tend to the deterministic profile $\eta_c(t_0 + T)$. This represents a wave structure of two successive wave crests lagged in time by T_2^* if

$$\begin{cases} \beta_0, \beta_1 \in \mathbf{R}_+^2 & \text{if } s \leqslant 0, \\ \beta_0, \beta_1 \in \Omega(s) & \text{if } s > 0 \end{cases} \tag{9}$$

as one can see from Fig. 3. Here, $\beta_0 = h_1/\sigma$, $\beta_1 = h_2/\sigma$ and $\Omega(s)$ is the open sectorial region of $\mathbf{R}_+^2$ with aperture angle $\theta = \pi/2 - 2\tan^{-1}(s)$, that is,

$$\Omega(s) = \left\{(\beta_0, \beta_1) \in \mathbf{R}_+^2 : \beta_0 \geqslant 0,\ \beta_1 \geqslant 0,\ s < \frac{\beta_1}{\beta_0} < \frac{1}{s}\right\}$$

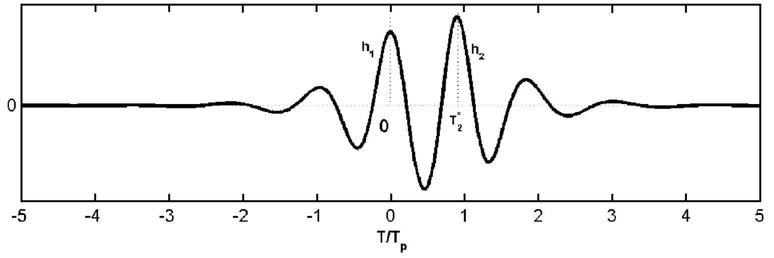

Fig. 3. Deterministic wave profile $\eta_c(T)$ of two successive wave crests.

and

$$s = \frac{\psi(T_2^*) + \ddot{\psi}(T_2^*)}{1 + \psi(T_2^*)\ddot{\psi}(T_2^*)}.$$

Typical JONSWAP spectrums satisfy the condition $s>0$ with $s \in [0.14, 0.16]$. As the spectrum gets narrow the sector $\Omega(s)$ tends to cover all $\mathbf{R}_+^2$, i.e. $\theta \to \pi/2$, because s approaches zero in the narrow-band limit.

In order to show that conditions (5) are also necessary, Fedele (2005) derived the analytical expression of the expected number per unit time $EX_c(\beta_0, \beta_1, \tau)$ of local maxima of the surface displacement $\eta(t)$ (at a fixed location in space) whose elevation is between β_0 and $\beta_0 + \mathrm{d}\beta_0$, and which are followed by a local maximum with an elevation between β_1 and $\beta_1 + \mathrm{d}\beta_1$ after a time lag between τ and $\tau + \mathrm{d}\tau$. Fedele showed that as both β_0 and $\beta_1 \to \infty$ in the τ-domain there exists an infinitesimal neighborhood $\delta\tau \sim O(\beta_0^{-1}, \beta_1^{-1})$ of $\tau = T_2^*$ such that

$$EX_c(\beta_0, \beta_1, \tau) = \begin{cases} EX_c(\beta_0, \beta_1, T_2^*) \exp(-\frac{1}{2} K^* \delta\tau^2), \\ 0 \quad \text{elsewhere}, \end{cases} \tag{10}$$

where the positive parameter $K^*>0$ is given by (the dot denotes time derivative)

$$K^* = -\frac{\ddot{\psi}(T_2^*)}{1 - \ddot{\psi}^2(T_2^*)} \ddot{\eta}_c(t_0)\ddot{\eta}_c(t_0 + T_2^*). \tag{11}$$

Here, $\ddot{\eta}_c(t_0)$ and $\ddot{\eta}_c(t_0 + T_2^*)$ are the second-order time derivatives of the deterministic profile $\eta_c(t_0 + T)$ evaluated at $T = 0$ and $T = T_2^*$, respectively, that is,

$$\ddot{\eta}_c(t_0) = a(-\beta_0 + s\beta_1), \quad \ddot{\eta}_c(t_0 + T_2^*) = a(-\beta_1 + s\beta_0), \tag{12}$$

where

$$a = \frac{1 + \psi(T_2^*)\ddot{\psi}(T_2^*)}{1 - \psi^2(T_2^*)}.$$

Note that K^* is greater or equal to zero because $\ddot{\psi}(T_2^*)<0$ by definition and both $\ddot{\eta}_c(t_0)<0$ and $\ddot{\eta}_c(t_0 + T_2^*)<0$ since $\eta_c(t_0 + T)$ has two local maxima at $t = t_0$ and

$t = t_0 + T_2^*$. Thus, two successive local maxima of dimensionless amplitude β_0 and β_1, respectively, attain the maximal expectation $EX_c(\beta_0, \beta_1, \tau)$ when the time lag between their occurrence is equal to $\tau = T_2^*$. Moreover, from Eq. (10) one concludes that $EX_c(\beta_0, \beta_1, T_2^* + \delta\tau) \simeq EX_c(\beta_0, \beta_1, T_2^*)$ for $\delta\tau$ of order $O(K_*^{-1/2})$ where $K_* \to \infty$ as both β_0 and $\beta_1 \to \infty$. This means that a local maximum of a very large amplitude β_0 followed by a local maximum of a very large amplitude β_1 after a time lag $T_2^* + \delta\tau$, with $\delta\tau \sim O(\beta_0^{-1}, \beta_1^{-1})$, has almost the same maximal expectation as two consecutive local maxima with amplitudes equal to β_0 and β_1, respectively, lagged in time by T_2^*. However, two local maxima of large amplitude lagged in time by T_2^* are also two successive crests because conditions (5) are sufficient. Hence, conditions (5) are also necessary in the limit of $\beta_0 \to \infty$ and $\beta_1 \to \infty$.

As a corollary, Fedele (2005) derived an upper bound for the joint probability density function of two successive wave crests as the following bivariate Rayleigh distribution:

$$p_W(\beta_0, \beta_1) = \frac{\beta_0 \beta_1}{(1-\theta^2)} \exp\left[-\frac{\beta_0^2 + \beta_1^2}{2(1-\theta^2)}\right] I_0\left(\frac{\theta \beta_0 \beta_1}{1-\theta^2}\right). \tag{13}$$

Here, the Rayleigh parameter is $\theta = \psi(T_2^*)/\psi(0)$. The bivariate Rayleigh distribution has been used by many authors to model the distribution of successive wave heights in narrow-band Gaussian seas (cf. Rodriguez et al., 2000; 2002) or the distribution of successive wave periods (cf. Myrhaug et al., 1995) and the parameter θ is estimated as

$$\theta_m = \sqrt{\frac{\left[\int_0^\infty S(\omega)\cos(\omega T_m)\,d\omega\right]^2 + \left[\int_0^\infty S(\omega)\sin(\omega T_m)\,d\omega\right]^2}{\sigma^2}} \tag{14}$$

with T_m the mean wave period. In Fig. 4 there are plotted the data points $(T_{sc}/T_2^*, (\beta_0 + \beta_1)/2)$ obtained from Monte Carlo simulations of Gaussian seas with mean JONSWAP spectrum (200 000 waves). Here, T_{sc}/T_2^* is the normalized period between two successive wave crests of dimensionless amplitudes β_0 and β_1, respectively. One can see that the period T_{sc}/T_2^* of the highest successive wave crests is very close to 1 in agreement with the theoretical results.

4. The occurrence of two successive wave crests in the space–time domain

In the following the analysis in the time domain of Fedele (2005) discussed in the preceding section is extended to the space–time domain. Consider the fixed point location $\mathbf{x} = \mathbf{x}_0$. What happens in the neighborhood of $\mathbf{x}_0$ if two large successive wave crests are recorded in time at $\mathbf{x}_0$? An infinite set of scenarios can be thought where collisions or interactions of different wave groups travelling with different speeds along different directions may cause the formation of two extreme consecutive crests. The space–time domain analysis presented here excludes all the

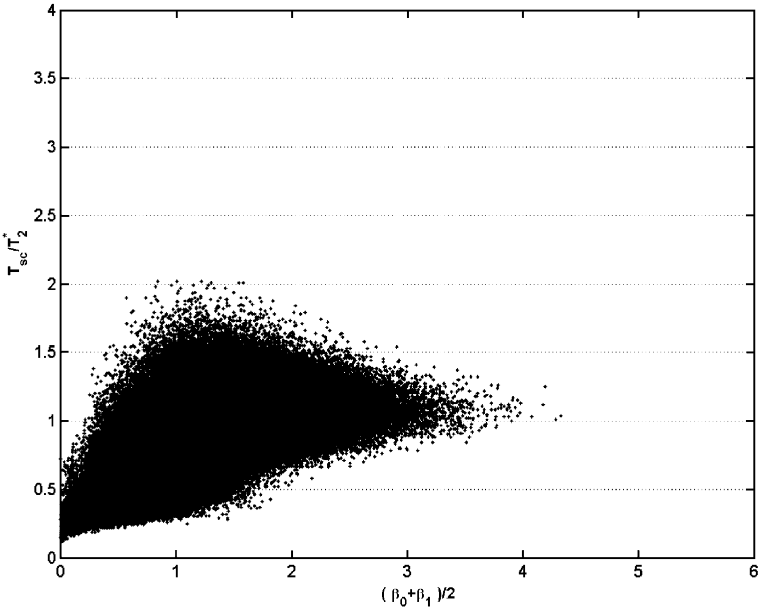

Fig. 4. Data points $(T_{sc}/T_2^*, (\beta_0 + \beta_1)/2)$ obtained from Monte Carlo simulations. Here, T_{sc} is the period between two successive wave crests of dimensionless amplitudes β_0 and β_1, respectively.

previous scenarios and surprisingly reveals that, with high probability, only a single wave group has passed through the point $\mathbf{x}_0$ in its initial phase of decaying when two large successive crests are recorded in the time domain at $\mathbf{x}_0$.

Consider the generic location $\mathbf{x} = \mathbf{x}_0 + \mathbf{X}$ where $\mathbf{X} = (X, Y)$. At $\mathbf{x} = \mathbf{x}_0$ it is known that the surface displacement $\xi(\mathbf{x}, t)$ has values equal to h_1 and h_2 at time $t = t_0$ and $t = t_0 + T_2^*$ respectively, that is,

$$\xi(\mathbf{x}_0, t_0) = h_1, \quad \xi(\mathbf{x}_0, t_0 + T_2^*) = h_2. \tag{15}$$

What is the probability that the surface displacement $\xi(\mathbf{x}, t)$ assumes values between u and $u + \mathrm{d}u$ at any fixed point $\mathbf{x} = \mathbf{x}_0 + \mathbf{X}$ at any time $t = t_0 + T$ if two successive wave crests of large amplitude are recorded at $\mathbf{x} = \mathbf{x}_0$ such as in Eq. (15)?

Consider the probability density function of the surface displacement $\xi(\mathbf{x}, t)$ at any fixed point $\mathbf{x} = \mathbf{x}_0 + \mathbf{X}$ at any time $t = t_0 + T$ given conditions (15), that is,

$$p[\xi(\mathbf{x}_0 + \mathbf{X}, t_0 + T) = u/\xi(\mathbf{x}_0, t_0) = h_1, \xi(\mathbf{x}_0, t_0 + T_2^*) = h_2].$$

This conditional p.d.f. is Gaussian and its conditional mean $\xi_c(\mathbf{x}_0 + \mathbf{X}, t_0 + T)$ is given by

$$\xi_c(\mathbf{x}_0 + \mathbf{X}, t_0 + T) = C_1 \Psi(\mathbf{X}, T) + C_2 \Psi(\mathbf{X}, T - T_2^*). \tag{16}$$

Here, the coefficients C_1 and C_2 are the same as in Eq. (8). The conditional space–time variance σ_c^2 admits the following expression

$$\frac{\sigma_c^2(\mathbf{x}_0+\mathbf{X},t_0+T)}{\sigma^2} = 1-\frac{\Psi^2(\mathbf{X},T)+\Psi(\mathbf{X},T-T_2^*)-2\Psi(\mathbf{X},T)\Psi(\mathbf{X},T-T_2^*)\Psi(\mathbf{0},T_2^*)/\Psi(\mathbf{0},0)}{\Psi(\mathbf{0},0)-\Psi^2(\mathbf{0},T_2^*)}. \tag{17}$$

In the limit of $h_1/\sigma\to\infty$ and $h_2/\sigma\to\infty$, the ratio $\sigma_c(\mathbf{x}_0+\mathbf{X},t_0+T)/\xi_c(\mathbf{x}_0+\mathbf{X},t_0+T)\to 0$ because $\xi_c(\mathbf{x}_0+\mathbf{X},t_0+T)$ tends to infinity and $\sigma_c(\mathbf{x}_0+\mathbf{X},t_0+T)$ is bounded by the unconditional standard deviation σ. Thus, all the realizations of the Gaussian sea satisfying conditions (5), with probability approaching 1, tend to the deterministic form $\xi_c(\mathbf{x}_0+\mathbf{X},t_0+T)$ for very large crest heights, that is,

$$p[\xi(\mathbf{x}_0+\mathbf{X},t_0+T)=u/\xi(\mathbf{x}_0,t_0)=h_1,\xi(\mathbf{x}_0,t_0+T_2^*)=h_2] \to \delta[u-\xi_c(\mathbf{x}_0+\mathbf{X},t_0+T)]$$

as both h_1/σ and $h_2/\sigma\to\infty$ where $\delta(x)$ is the Dirac function. In the space–time domain, the conditional mean $\xi_c(\mathbf{x}_0+\mathbf{X},t_0+T)$ (see Eq. (16)) represents a wave group which has passed closed by $\mathbf{x}_0$ at the apex of its development stage. It is during its initial phase of decaying that at the location $\mathbf{X}=\mathbf{0}$, i.e. $\mathbf{x}=\mathbf{x}_0$, two successive wave crests lagged in time by T_2^* occur. In fact, at $\mathbf{X}=\mathbf{0}$, the time series $\xi_c(\mathbf{x}_0,t_0+T)$ is the same as the conditional mean $\eta_c(t_0+T)$ in Eq. (7). As an application, consider the case of unidirectional waves in deep water with a mean JONSWAP spectrum. According to Eq. (3), the wave group (16) can be written as

$$\xi_c(X,T)=\int_0^\infty \tilde{S}(\omega)\cos(kX-\omega T+\phi(\omega))\,\mathrm{d}\omega, \tag{18}$$

where the spectrum $\tilde{S}(\omega)$ and phase function $\phi(\omega)$ are given by

$$\tilde{S}(\omega)=S(\omega)\sqrt{C_1^2+C_2^2+2C_1C_2\cos(\omega T_2^*)}, \tag{19}$$

$$\phi(\omega)=\arctan\frac{C_2\sin(\omega T_2^*)}{C_1+C_2\cos(\omega T_2^*)}$$

and the wave number $k=\omega^2/g$. For sake of simplicity, assume a mean JONSWAP spectrum ($T_2^*\simeq 0.9T_\mathrm{p}$) and equal amplitudes $h_1=h_2=h$. In Fig. 5 the snapshots of the wave group computed numerically using Eq. (16) at successive times T are plotted. As one can see, the wave group moves in deep water with the group velocity $c_\mathrm{g}\simeq 0.5L_\mathrm{p}/T_\mathrm{p}$. A contraction of the group occurs as time evolves and the max development stage is reached at time $T\simeq -0.5T_2^*=-0.45T_\mathrm{p}$ at the location $x=x_0-0.41L_\mathrm{p}$, i.e. $X\simeq -0.41L_\mathrm{p}$, and gives rise to the isolated highest crest of amplitude $H_{\max}=1.1h$. Subsequently, the group tends to expand and decay. It is during this initial phase of decaying of the wave group that the two successive wave crests are formed at time $T=0$ and $T=T_2^*$ at the location $x=x_0$, i.e. $X=0$. In order to explain this dynamics of the wave group, in Fig. 5 at time $T=-0.5T_2^*$,

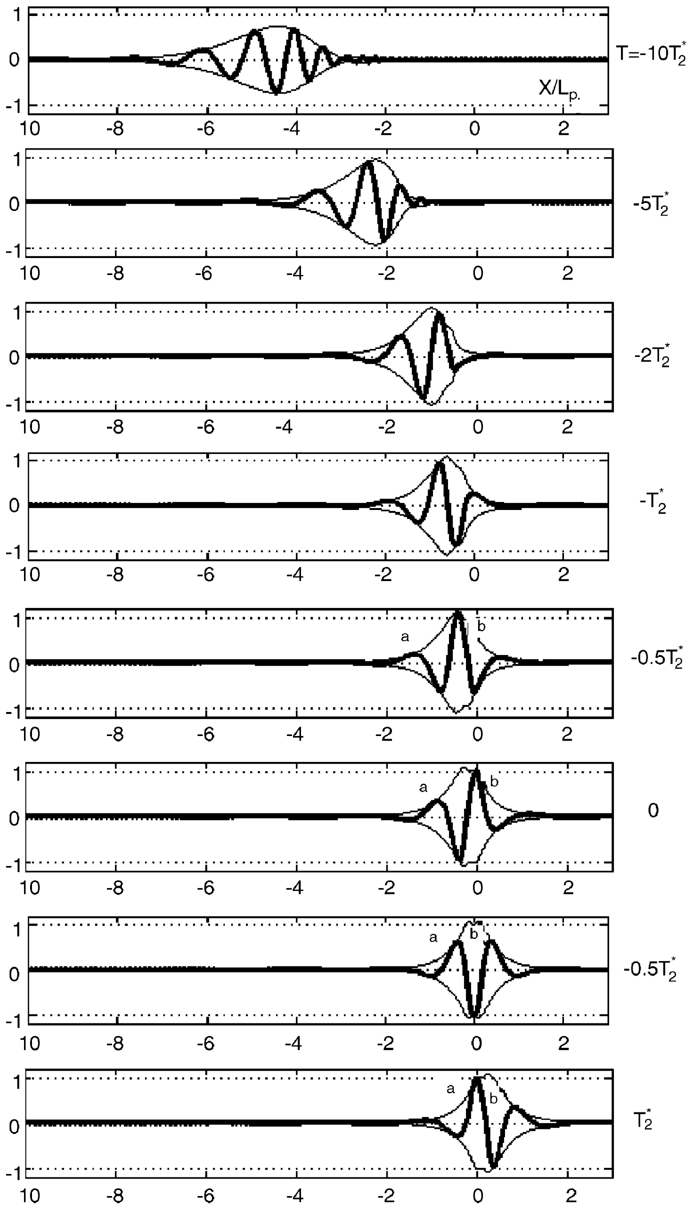

Fig. 5. Snapshots of the wave group in Eq. (16) for $h_1 = h_2 = h$. The y-axis represents surface amplitudes normalized with respect to h.

consider the two waves labelled as a and b, respectively. At this time the wave b is the central wave of the group with the max amplitude $H_{\max} = 1.1h$, whereas the wave a is just born from the tail of the group. As time evolves the wave a grows and the wave b decreases in amplitude so that after a certain point in time the wave a tends to

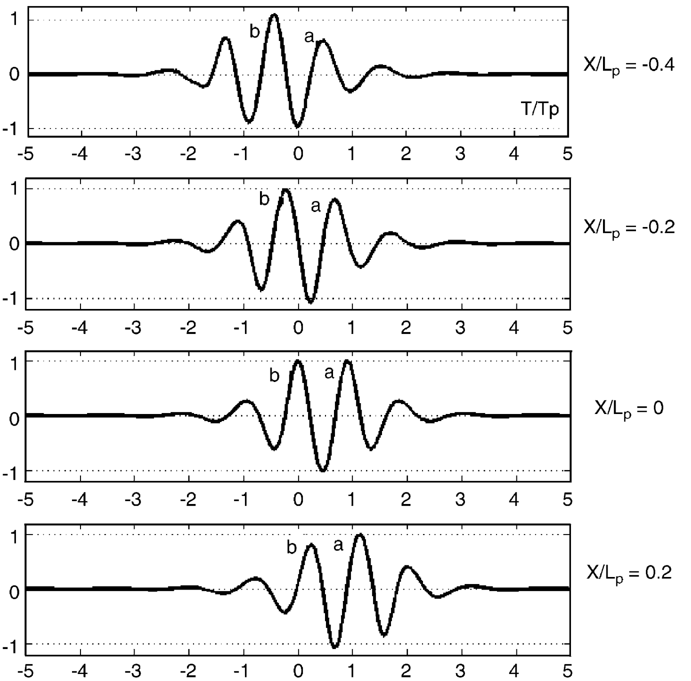

Fig. 6. Time series of the wave group in Eq. (16) at different X locations for $h_1 = h_2 = h$. The y-axis represents surface amplitudes normalized with respect to h.

become the central wave of the group and the wave b tends to disappear from the main core of the group as one can see from the snapshot at time $T = T_2^*$ in Fig. 5. Thus wave b interchanges its role of central wave of the group with wave a as the wave group evolves in time. It is during this interchange event that two successive wave crests of equal amplitude h occur at $X = 0$. In particular, the first crest, which occurs at time $T = 0$, is due to the relaxation of the wave b which has reached its max amplitude at time $T = -0.5T_2^*$, whereas the second crest occurs at time $T = T_2^*$ because of the growth of the wave a. The interchange of roles between waves a and b is also clear from Fig. 6 where the time series of the wave group at different space locations are plotted. In particular, at $X = -0.4L_p$ the wave b is at the apex of its development stage with the max amplitude $H_{max} = 1.1h$. As time evolves the group moves along the X-axis and the wave b tends to decrease in amplitude, whereas the wave a grows in amplitude such that at $X = 0$ they both occur with the same amplitude h forming two successive crests. Similar conclusions also hold for three-dimensional waves in undisturbed sea.

5. Conclusions

The theory of quasi-determinism of Boccotti (2000) is extended to investigate the occurrence of successive wave crests in a Gaussian sea. It is proven that if two consecutive exceptionally high wave crests are recorded at a point during a sea storm, most probably, a well defined wave group has transited at that point in the early stage of its decay. Boccotti's theory then reveals that the single wave group can be thought as a 'gene' for a Gaussian sea when the interest is in the dynamics of the surface displacement at high amplitude levels. With high probability the occurrence of extreme events in a Gaussian sea are due to the dynamics of a wave group: either an isolated extreme crest, an extreme crest-to-trough wave height or two consecutive extreme crests occur for different configurations of the same wave group. The relevance of this result is twofold. Firstly, the design of offshore structures resisting to a double wave impact can be based on the wave forces generated by the mechanics of a single wave group. On the other hand, this analysis provides a mathematical mean for studying extreme events and their probability of occurrence in the context of linear and nonlinear water waves by investigating the nonlinear evolution of a wave group (cf. Fedele, 2004, 2006; Fedele and Arena, 2005).

Appendix

According to the theory of sea states, to the first order in a Stokes expansion, a unidirectional random wave field can be represented as the linear superimposition of a large number of wave harmonics as

$$\xi(x,t)=\sum_{i=1}^{N} a_i\cos(k_i x-\omega_i t+\varepsilon_i). \tag{20}$$

Here, it is assumed that frequencies ω_i are different from each other, the number N is infinitely large and the phase angles ε_i, uniformly distributed in $[0,2\pi]$, are stochastically independent of each other. Furthermore, all the amplitudes a_i satisfy the frequency spectrum $S(\omega)$ defined as

$$S(\omega)\Delta\omega=\sum_i \frac{a_i^2}{2},\quad \omega_i\in\left(\omega-\frac{\Delta\omega}{2},\omega+\frac{\Delta\omega}{2}\right). \tag{21}$$

The jth order moment of the spectrum is $m_j=\int_0^\infty \omega^j S(\omega)\,\mathrm{d}\omega$. In particular, $m_0=\sigma^2$, where σ is the standard deviation of η. For a fixed point $x=x_0$ at the sea, the recorded time series $\eta(t)=\xi(x_0,t)$ of the surface displacement $\xi(x,t)$ is given by

$$\eta(t)=\sum_{i=1}^{N} a_i\cos(\omega_i t+\tilde{\varepsilon}_i), \tag{22}$$

where $\tilde{\varepsilon}_i=\mathrm{mod}(k_i x_0+\varepsilon_i,2\pi)$. In the limit of $N\to\infty$, $\eta(t)$ is a realization of a stationary ergodic stochastic Gaussian process.

References

Arena, F., Fedele, F., 2005. Non-linear space–time evolution of a high wave crest. ASME Journal of Offshore Mechanics and Arctic Engineering 127 (1), 1–74.

Boccotti, P., 1981. On the highest waves in a stationary Gaussian process. Atti Accademia Ligure di Scienze e Lettere 38, 271–302.
Boccotti, P., 1982. On ocean waves with high crests. Meccanica 17, 16–19.
Boccotti, P., 1983. Some new results on statistical properties of wind waves. Applied Ocean Research 5, 134–140.
Boccotti, P., 1989. On mechanics of irregular gravity waves. Atti Accademia Nazionale Lincei, Memorie 19, 11–170.
Boccotti, P., 1997. A general theory of three-dimensional wave groups. Ocean Engineering 24, 265–300.
Boccotti, P., 2000. Wave Mechanics for Ocean Engineering. Elsevier, Oxford.
Boccotti, P., Barbaro, G., Mannino, L., 1993a. A field experiment on the mechanics of irregular gravity waves. Journal of Fluid Mechanics 252, 173–186.
Boccotti, P., Barbaro, G., Fiamma, V., et al., 1993b. An experiment at sea on the reflection of the wind waves. Ocean Engineering 20, 493–507.
Fedele, F., 2004. The occurrence of extreme crests and the nonlinear wave–wave interaction in random seas. Proceedings of XIV International Offshore and Polar Engineering Conference (ISOPE), Toulon, France, vol. III, pp. 55–63.
Fedele, F., 2005. Successive wave crests in Gaussian seas. Probabilistic Engineering Mechanics 20 (4), 355–363.
Fedele, F., 2006. Extreme events in nonlinear random seas. ASME Journal of Offshore Mechanics and Arctic Engineering 128 (1), 1–6.
Fedele, F., Arena, F., 2005. Weakly nonlinear statistics of high random waves. Physics of fluids 17(1), 026601, 1–10.
Hasselmann, K., Barnett, T.P., Bouws, E., et al., 1973. Measurements of wind wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP). Deutsche Hydrographische Zeitschrift A8, 1–95.
Lindgren, G., 1970. Some properties of a normal process near a local maximum. Annals of Mathematical Statistics 4 (6), 1870–1883.
Lindgren, G., 1972. Local maxima of Gaussian fields. Arkiv for Matematik 10, 195–218.
Longuet-Higgins, M.S., 1952. On the statistical distribution of the heights of sea waves. Journal of Marine Research 11, 245–266.
Longuet-Higgins, M.S., 1980. On the distribution of the heights of sea waves; some effects of nonlinearity and finite band width. Journal of Geophysical Research 85 (C8), 1519–1528.
Maes, M.A., Breitung, K.W., 1997. Direct Approximation of the extreme value distribution of nonhomogeneous Gaussian random fields. ASME Journal of Offshore Mechanics and Arctic Engineering 119, 252–256.
Myrhaug, D., et al., 1995. A two dimensional Weibull distribution and its application to rolling. ASME Journal of Offshore Mechanics and Arctic Engineering 117 (3), 178–182.
Naess, A., 1985. On the statistical distribution of crest to trough wave heights. Ocean Engineering 12, 221–234.
Phillips, O.M., Gu, D., Donelan, M., 1993a. On the expected structure of extreme waves in a Gaussian sea, I. Theory and SWADE buoy measurements. Journal of Physical Oceanography 23, 992–1000.
Phillips, O.M., Gu, D., Walsh, E.J., 1993b. On the expected structure of extreme waves in a Gaussian sea, II. SWADE scanning radar altimeter measurements. Journal of Physical Oceanography 23, 2297–2309.
Rodriguez, G., Soares, C.G., et al., 2000. Wave groups statistics of numerically simulated mixed sea states. ASME Journal of Offshore Mechanics and Arctic Engineering 122 (11), 282–288.
Rodriguez, G., Soares, C.G., et al., 2002. Wave height distribution in mixed sea states. ASME Journal of Offshore Mechanics and Arctic Engineering 124 (2), 34–39.
Sun, J., 1993. Tail probabilities of the maxima of Gaussian random fields. The Annals of Probability 21 (1), 34–71.
Tayfun, M.A., 1981. Distribution of crest-to-trough wave heights. ASCE Journal of Waterway, Port, Coastal and Ocean Engineering 107 (WW3), 149–156.
Tayfun, M.A., 1990. Distribution of large wave heights. ASCE Journal of Waterway, Port, Coastal and Ocean Engineering 116 (6), 686–707.
Tromans, P.S., Anaturk, A.R., Hagemeijer, P., 1991. A new model for the kinematics of large ocean waves—application as a design wave. Shell International Research 1042.